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Alexander Ziwet

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AN INTRODUCTION TO THE USE OF GENERALIZED COÖRDINATES IN MECHANICS AND PHYSICS

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PREFACE

This book was undertaken at the suggestion of my lamented colleague Professor Benjamin Osgood Peirce, and with the promise of his collaboration. His untimely death deprived me of his invaluable assistance while the second chapter of the work was still unfinished, and I have been obliged to complete my task without the aid of his remarkably wide and accurate knowledge of Mathematical Physics.

The books to which I am most indebted in preparing this treatise are Thomson and Tait's "Treatise on Natural Philosophy," Watson and Burbury's "Generalized Coördinates," Clerk Maxwell's "Electricity and Magnetism," E. J. Routh's "Dynamics of a Rigid Body," A. G. Webster's "Dynamics," and E. B. Wilson's "Advanced Calculus."

For their kindness in reading and criticizing my manuscript I am indebted to my friends Professor Arthur Gordon Webster, Professor Percy Bridgman, and Professor Harvey Newton Davis.

W. E. BYERLY

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GENERALIZED COÖRDINATES

CHAPTER I

INTRODUCTION

1. Coördinates of a Point. The position of a moving particle may be given at any time by giving its rectangular coördinates x, y, z referred to a set of rectangular axes fixed in space. It may be given equally well by giving the values of any three specified functions of x, y, and z, if from the values in question the corresponding values of x, y, and z may be obtained uniquely. These functions may be used as coördinates of the point, and the values of x, y, and z expressed explicitly in terms of them serve as formulas for transformation from the rectangular system to the new system. Familiar examples are *polar* coördinates in a plane, and *cylindrical* and *spherical* coördinates in space, the formulas for transformation of coördinates being respectively

$$\left.\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \end{aligned}\right\} \quad (1) \qquad \left.\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \\ z &= z, \end{aligned}\right\} \quad (2) \qquad \left.\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta\cos\phi, \\ z &= r\sin\theta\sin\phi. \end{aligned}\right\} \quad (3)$$

It is clear that the number of possible systems of coördinates is unlimited. It is also clear that if the point is unrestricted in its motion, three coördinates are required to determine it. If it is restricted to moving in a plane, since that plane may be taken as one of the rectangular coördinate planes, two coördinates are required.

The number of independent coördinates required to fix the position of a particle moving under any given conditions is called the number of *degrees of freedom* of the particle, and is

equal to the number of independent conditions required to fix the point.

Obviously these coördinates must be numerous enough to fix the position without ambiguity and not so numerous as to render it impossible to change any one at pleasure without changing any of the others and without violating the restrictions of the problem.

2. Dynamics of a Particle. *Free Motion.* The differential equations for the motion of a particle under any forces when we use rectangular coördinates are known to be

$$\left.\begin{aligned} m\ddot{x} &= X, \\ m\ddot{y} &= Y, \\ m\ddot{z} &= Z. \end{aligned}\right\} \tag{1}^*$$

X, Y, and Z, the components of the actual forces on the particle resolved parallel to the fixed rectangular axes, or rather their equivalents $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$, are called the *effective forces* on the particle. They are of course a set of forces mechanically equivalent to the actual forces acting on the particle.

The equations of motion of the particle in terms of any other system of coördinates are easily obtained.

Let q_1, q_2, q_3, be the coördinates in question. The appropriate formulas for transformation of coördinates express x, y, and z in terms of q_1, q_2, and q_3.

$$x = f_1(q_1, q_2, q_3), \qquad y = f_2(q_1, q_2, q_3), \qquad z = f_3(q_1, q_2, q_3).$$

For the component velocity $\dot{x}$ we have

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3,$$

and $\dot{x}$, $\dot{y}$, and $\dot{z}$ are explicit functions of q_1, q_2, q_3, $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$, linear and homogeneous in terms of $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$.

* For time derivatives we shall use the Newtonian *fluxion* notation, so that we shall write $\dot{x}$ for $\frac{dx}{dt}$, $\ddot{x}$ for $\frac{d^2x}{dt^2}$.

We may note in passing that it follows from this fact that $\dot{x}^2$, $\dot{y}^2$, and $\dot{z}^2$ are homogeneous quadratic functions of $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$.

Obviously
$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}; \tag{2}$$

and since
$$\frac{d}{dt}\frac{\partial x}{\partial q_1} = \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_2 + \frac{\partial^2 x}{\partial q_3 \partial q_1}\dot{q}_3,$$

and
$$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2 + \frac{\partial^2 x}{\partial q_1 \partial q_3}\dot{q}_3,$$

$$\frac{d}{dt}\frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial q_1}. \tag{3}$$

Let us find now an expression for the work $\delta_{q_1}W$ done by the effective forces when the coördinate q_1 is changed by an infinitesimal amount δq_1 without changing q_2 or q_3. If δx, δy, and δz are the changes thus produced in x, y, and z, obviously

$$\delta_{q_1}W = m\left[\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z\right]$$

if expressed in rectangular coördinates. We need, however, to express $\delta_{q_1}W$ in terms of our coördinates q_1, q_2, and q_3.

$$\delta_{q_1}W = m\left[\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right]\delta q_1.$$

Now
$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{d}{dt}\frac{\partial x}{\partial q_1};$$

but by (2) and (3)

$$\frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial \dot{q}_1} \quad \text{and} \quad \frac{d}{dt}\frac{\partial x}{\partial q_1} = \frac{\partial \dot{x}}{\partial q_1}.$$

Hence
$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right) - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right);$$

and therefore
$$\delta_{q_1}W = \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1}\right]\delta q_1, \tag{4}$$

where
$$T = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]$$

and is the *kinetic energy* of the particle.

To get our differential equation we have only to write the second member of (4) equal to the work done by the actual forces when q_1 is changed by δq_1.

If we represent the work in question by $Q_1\delta q_1$, our equation is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1, \tag{5}$$

and of course we get such an equation for every coördinate.

It must be noted that usually equation (5) will contain q_2 and q_3 and their time derivatives as well as q_1, and therefore cannot be solved without the aid of the other equations of the set.

In any concrete problem, T must be expressed in terms of q_1, q_2, q_3, and their time derivatives before we can form the expression for the work done by the effective forces. $Q_1\delta q_1$, $Q_2\delta q_2$, $Q_3\delta q_3$, the work done by the actual forces, must be obtained from direct examination of the problem.

3. (*a*) As an example let us get the equations in polar coördinates for motion in a plane.

Here $$x = r\cos\phi, \qquad y = r\sin\phi.$$

$$\dot{x}^2 + \dot{y}^2 = v^2 = \dot{r}^2 + r^2\dot{\phi}^2,$$

and $$T = \frac{m}{2}[\dot{r}^2 + r^2\dot{\phi}^2].$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial T}{\partial r} = mr\dot{\phi}^2.$$

$$\delta_r W = m[\ddot{r} - r\dot{\phi}^2]\,\delta r = R\delta r$$

if R is the impressed force resolved along the radius vector.

$$\frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi},$$

$$\frac{\partial T}{\partial \phi} = 0.$$

$$\delta_\phi W = m\frac{d}{dt}(r^2\dot{\phi})\,\delta\phi = \Phi r\delta\phi$$

if Φ is the impressed force resolved perpendicular to the radius vector.

In more familiar form

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right] = R,$$

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = \Phi.$$

(*b*) In cylindrical coördinates where

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z,$$

$$T = \frac{m}{2}[\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2].$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial T}{\partial r} = mr\dot{\phi}^2,$$

$$\frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi},$$

$$\frac{\partial T}{\partial \dot{z}} = m\dot{z}.$$

$$\delta_r W = m[\ddot{r} - r\dot{\phi}^2]\,\delta r = R\delta r,$$

$$\delta_\phi W = m\frac{d}{dt}(r^2\dot{\phi})\,\delta\phi = \Phi r\delta\phi,$$

$$\delta_z W = m\ddot{z}\delta z = Z\delta z;$$

or

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right] = R,$$

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = \Phi,$$

$$m\frac{d^2z}{dt^2} = Z.$$

$$\frac{1}{\dot{\theta}} = \frac{ma + kVat}{mV},$$

$$\frac{d\theta}{dt} = \frac{mV}{ma + kVat},$$

$$\theta = \frac{m}{ka} \log [m + kVt] + C,$$

$$\theta = \frac{m}{ka} \log \left[1 + \frac{kVt}{m}\right]; \qquad (1)$$

and the problem of the motion is completely solved.

(*b*) If, however, we are interested in R, the pressure of the constraining curve, we must proceed somewhat differently. We have only to replace the constraint by a force R directed toward the center of the path. There are now two degrees of freedom, and we shall take θ and the radius vector r as our coördinates and form two differential equations of motion.

$$T = \frac{m}{2} [\dot{r}^2 + r^2\dot{\theta}^2].$$

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2.$$

$$m\frac{d}{dt}(r^2\dot{\theta})\,\delta\theta = -kr^3\dot{\theta}^2\delta\theta. \qquad (1)$$

$$m(\ddot{r} - r\dot{\theta}^2)\,\delta r = -R\delta r. \qquad (2)$$

To these we may add

$$r = a.$$

Whence $$\ddot{\theta} + \frac{ka}{m}\dot{\theta}^2 = 0, \text{ as before,} \qquad (3)$$

and $$R = ma\dot{\theta}^2. \qquad (4)$$

(c) Let us now suppose that the constraining circle is rough. Here, since the friction is μ (the coefficient of friction) multiplied by the normal pressure, R will be needed, and we must replace the constraint by R as before.

We have now

$$m\frac{d}{dt}(r^2\dot{\theta})\,\delta\theta = -kr^3\dot{\theta}^2\delta\theta - \mu Rr\delta\theta,$$

$$m[\ddot{r} - r\dot{\theta}^2]\,\delta r = -R\delta r,$$

and $$r = a.$$

Whence $$R = ma\dot{\theta}^2, \text{ as before.}$$

$$\ddot{\theta} + \frac{ka}{m}\dot{\theta}^2 + \frac{\mu}{ma}R = 0,$$

$$\ddot{\theta} + \left[\frac{ka}{m} + \mu\right]\dot{\theta}^2 = 0.$$

Replacing $\frac{ka}{m}$ in Art. 5, (a), (1), by $\frac{ka}{m} + \mu$,

we have $$\theta = \frac{1}{\frac{ka}{m} + \mu}\log\left[1 + \left(\frac{ka}{m} + \mu\right)\frac{Vt}{a}\right]. \qquad (1)$$

EXAMPLES

1. Obtain the familiar equation $\frac{d^2\theta}{dt^2} + \frac{g}{a}\sin\theta = 0$ for the simple pendulum.

2. Find the tension of the string in the simple pendulum.

Ans. $R = m\left[g\cos\theta + a\left(\frac{d\theta}{dt}\right)^2\right]$.

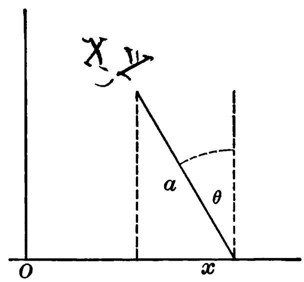
3. Obtain the equations of the spherical pendulum in terms of the spherical coördinates θ and ϕ.

Ans. $\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{g}{a}\sin\theta = 0. \qquad \sin^2\theta\dot{\phi} = C.$

6. (a) The constraint may not be so simple as that imposed by compelling the moving particle to remain on a given surface or on a given curve.

Take, for example, the *tractrix* problem, when the particle moves on a smooth horizontal plane.

Let a particle of mass m, attached to a string of length a, rest on a smooth horizontal plane. The string lies straight on the plane at the start, and then the end not attached to the particle is drawn with uniform velocity along a straight line perpendicular to the initial position of the string and lying in the plane. Let us take as our coördinates x, the distance traveled by that end of the string which is not attached to the particle, and θ, the angle made by the string with its initial position. Let R be the tension of the string and n the velocity with which the end of the string is drawn along. Let X, Y, be the rectangular coördinates of the particle, referred to the fixed line and to the initial position of the string as axes.



$$X = x - a\sin\theta,$$

$$Y = a\cos\theta;$$

$$\dot{X} = \dot{x} - a\cos\theta\dot{\theta};$$

$$\dot{Y} = -a\sin\theta\dot{\theta}.$$

$$T = \frac{m}{2}(\dot{X}^2 + \dot{Y}^2) = \frac{m}{2}[\dot{x}^2 + a^2\dot{\theta}^2 - 2a\cos\theta\dot{x}\dot{\theta}].$$

$$\frac{\partial T}{\partial \dot{x}} = m[\dot{x} - a\cos\theta\dot{\theta}],$$

$$\frac{\partial T}{\partial \dot{\theta}} = m[a^2\dot{\theta} - a\cos\theta\dot{x}],$$

$$\frac{\partial T}{\partial \theta} = ma\sin\theta\dot{x}\dot{\theta}.$$

$$m\frac{d}{dt}[\dot{x} - a\cos\theta\dot{\theta}]\,\delta x = R\sin\theta\delta x.$$

$$m\left[\frac{d}{dt}(a^2\dot{\theta} - a\cos\theta\dot{x}) - a\sin\theta\dot{x}\dot{\theta}\right]\delta\theta = 0.$$

Adding the condition $x = nt,$

and reducing, $-ma[\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2] = R\sin\theta.$

$$ma^2\ddot{\theta} = 0.$$

$$\ddot{\theta} = 0.$$

$$R = ma\dot{\theta}^2.$$

Integrating, $\dot{\theta} = C = \frac{n}{a}.$

$$R = \frac{mn^2}{a}.$$

The particle revolves with uniform angular velocity about the moving center, and the pull on the string is constant.

(b) A particle is at rest in a smooth horizontal tube. The tube is then made to revolve in a horizontal plane with uniform angular velocity ω. Find the motion of the particle.

Suggestion. Take the polar coördinates r, ϕ, of the particle as our coördinates, and let R be the pressure of the particle on the tube.

$$T = \frac{m}{2}[\dot{r}^2 + r^2\dot{\phi}^2].$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial T}{\partial r} = mr\dot{\phi}^2,$$

$$\frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi}.$$

$$m[\ddot{r} - r\dot{\phi}^2]\delta r = 0.$$

$$m\frac{d}{dt}(r^2\dot{\phi})\delta\phi = Rr\delta\phi.$$

Adding the condition $\phi = \omega t,$

and reducing, $\ddot{r} - \omega^2 r = 0,$

$$2m\omega r\dot{r} = Rr.$$

Solving, $$r = A \cosh \omega t + B \sinh \omega t,$$

$$r = a \quad \text{and} \quad \dot{r} = 0 \text{ at the start.}$$

Hence $$r = a \cosh \omega t = a \cosh \phi,$$

$$R = 2\, ma\omega^2 \sinh \omega t = 2\, ma\omega^2 \sinh \phi.$$

If we are interested only in the motions and not in the reactions, problems (a) and (b) can be solved more simply. If in each we were to use one less coördinate, θ only in (a) and r only in (b), rectangular coördinates X, Y, for the particle could be obtained whenever the time was given, and therefore could be expressed explicitly in terms of θ or r and t. A careful examination of Art. 2 will show that the reasoning is extended easily to such a case, and that the work done by the effective forces when q_1 only is changed is still $\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1}\right]\delta q_1$. It is to be noted, however, that when the rectangular coördinates are functions of t as well as of q_1, q_2, etc., the energy T is no longer a homogeneous quadratic in $\dot{q}_1$, $\dot{q}_2$, etc.

(a') $$X = nt - a \sin \theta,$$

$$Y = a \cos \theta;$$

$$\dot{X} = n - a \cos \theta \dot{\theta},$$

$$\dot{Y} = - a \sin \theta \dot{\theta}.$$

$$T = \frac{m}{2}[\dot{X}^2 + \dot{Y}^2] = \frac{m}{2}[n^2 + a^2\dot{\theta}^2 - 2\, an \cos \theta \dot{\theta}].$$

$$\frac{\partial T}{\partial \dot{\theta}} = m\,[a^2\dot{\theta} - an \cos \theta],$$

$$\frac{\partial T}{\partial \theta} = man \sin \theta \dot{\theta}.$$

$$m\left[\frac{dt}{d}(a^2\dot{\theta} - an \cos \theta) - an \sin \theta \dot{\theta}\right]\delta\theta = 0.$$

$$\ddot{\theta} = 0.$$

$$\dot{\theta} = \frac{n}{a}, \text{ as before.}$$

$$(b') \qquad T = \frac{m}{2}[\dot{r}^2 + \omega^2 r^2].$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial T}{\partial r} = m\omega^2 r.$$

$$m[\ddot{r} - \omega^2 r]\,\delta r = 0.$$

$$r = a \cosh \omega t, \text{ as before.}$$

EXAMPLES

1. A particle rests on a smooth horizontal whirling table and is attached by a string of length a to a point fixed in the table at a distance b from the center. The particle, the point, and the center are initially in the same straight line. The table is then made to rotate with uniform angular velocity ω. Find the motion of the particle.

Suggestion. Take as the single coördinate θ the angle made by the string with the radius of the point. Let X, Y, be the rectangular coördinates of the particle, referred to the line initially joining it with the center and to a perpendicular thereto through the center as axes.

Then $$X = b \cos \omega t + a \cos(\theta + \omega t),$$

and $$Y = b \sin \omega t + a \sin(\theta + \omega t).$$

$$T = \frac{m}{2}[b^2\omega^2 + a^2(\omega + \dot{\theta})^2 + 2\,ab\omega \cos\theta\,(\omega + \dot{\theta})],$$

and $$\ddot{\theta} + \frac{b\omega^2}{a}\sin\theta = 0;$$

and the relative motion on the table is simple pendulum motion, the length of the equivalent pendulum being $l = \dfrac{ag}{b\omega^2}$.

2. A particle is attracted toward a fixed point in a horizontal whirling table with a force proportional to the distance. It is initially at rest at the center. The table is then made to rotate

with uniform angular velocity ω. Find the path traced on the table by the particle.

Suggestion. Take as coördinates x, y, rectangular coördinates referred to the moving radius of the fixed point as axis of abscissas and to the center of the table as origin. Let X, Y, be the rectangular coördinates referred to fixed axes coinciding with the initial positions of the moving axes.

$$X = x\cos\omega t - y\sin\omega t, \qquad Y = x\sin\omega t + y\cos\omega t.$$

$$T = \frac{m}{2}[\dot{x}^2 + \omega^2x^2 + \dot{y}^2 + \omega^2y^2 - 2\,\omega y\dot{x} + 2\,\omega x\dot{y}].$$

Whence come $m[\ddot{x} - 2\,\omega\dot{y} - \omega^2x] = -\mu(x-a)$,

$$m[\ddot{y} + 2\,\omega\dot{x} - \omega^2y] = -\mu y.$$

If $\omega^2 = \frac{\mu}{m}$, the solution is easy and interesting.

$$\ddot{x} - 2\,\omega\dot{y} = a\omega^2, \tag{1}$$

$$\ddot{y} + 2\,\omega\dot{x} = 0. \tag{2}$$

Integrating (2), $\dot{y} + 2\,\omega x = 0.$

Substituting in (1), $\ddot{x} + 4\,\omega^2x = a\omega^2. \quad (3)$

Multiplying (3) by $2\,\dot{x}$, and integrating,

$$\dot{x}^2 + 4\,\omega^2x^2 = 2\,a\omega^2x.$$

$$\dot{x} = 2\,\omega\sqrt{\frac{ax}{2} - x^2}.$$

Whence $\qquad 2\,\omega t = \text{vers}^{-1}\dfrac{4\,x}{a}.$

$$x = \frac{a}{4}[1 - \cos 2\,\omega t],$$

$$y = -\frac{a}{4}[2\,\omega t - \sin 2\,\omega t].$$

Replacing $2\,\omega t$ by θ, $x = \dfrac{a}{4}[1 - \cos\theta]$,

$$y = -\frac{a}{4}[\theta - \sin\theta],$$

and the curve traced on the table is the cycloid generated by a circle of radius $\frac{a}{4}$ rolling backward along the moving axis of Y.

7. A System of Particles. If instead of a single particle we have a system of particles, free, or connected or otherwise constrained, $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$, are the effective forces on the particle P. The effective forces on all the particles are spoken of as the effective forces on the system and are mechanically equivalent to the set of actual forces on the system.

T, the *kinetic energy* of the system, is the sum of the kinetic energies of all the particles.

$$T = \sum \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2].$$

If δW is the work done by the effective forces in any supposed infinitesimal displacement of the system,

$$\delta W = \Sigma m[\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z].$$

If the particles of a moving system are subjected to connections or constraints, these connections or constraints may or may not vary with the time. In the latter case a set of any n independent variables $q_1, q_2, \cdots, q_n$, such that when they and the connections and constraints are given, the position of every particle of the system is uniquely determined, and such that when the positions of all the particles of the system are given, $q_1, q_2, \cdots, q_n$, follow uniquely, may be taken as coördinates of the system; and n is called the *number of degrees of freedom* of the system.

In the former case a set of variables $q_1, q_2, \cdots, q_n$, such that when they and the time are given, the position of every particle of the system is uniquely determined, and such that when the positions of all the particles of the system and the time are given, $q_1, q_2, \cdots, q_n$, follow uniquely, may be taken as the coördinates of the system; and n is called the number of degrees of freedom of the system.

The equations expressing the connections and constraints in terms of the rectangular coördinates of the particles and of the coördinates $q_1, q_2, \cdots, q_n$, of the system are often called the *geometrical equations* of the system and may or may not contain the time explicitly. In the latter case the geometrical equations make it possible to express the coördinates x, y, z, of every point of the system explicitly as functions of the q's; in the former case, as functions of t and the q's.

The geometrical equations must not contain explicitly either the time derivatives of the rectangular coördinates of the particles or those of the coördinates $q_1, q_2, \cdots, q_n$, of the system unless they can be freed from these derivatives by integration.

Examples of geometrical equations not containing the time explicitly are the formulas for transformation of coördinates in Arts. 1 and 3, and the equations for X and Y in Art. 6, (a).

Geometrical equations containing the time are the equations for X and Y in Art. 6, (a'), and in Art. 6, Exs. 1 and 2.

The work, $\delta_{q_1}W$, done by the effective forces when q_1 is changed by δq_1 without changing the other q's is proved to be equal to

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1}\right]\delta q_1$$

by reasoning similar to that used in Art. 2. For the sake of variety we take the case where the geometrical equations involve the time.

Here $$x = f[t, q_1, q_2, \cdots, q_n].$$

$$\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial x}{\partial q_n}\dot{q}_n,$$

and is an explicit function of $t, q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$.

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}, \tag{1}$$

and since $\dfrac{d}{dt}\dfrac{\partial x}{\partial q_1} = \dfrac{\partial^2 x}{\partial t \partial q_1} + \dfrac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \dfrac{\partial^2 x}{\partial q_2 \partial q_1}\dot{q}_2 + \cdots + \dfrac{\partial^2 x}{\partial q_n \partial q_1}\dot{q}_n,$

and $$\frac{\partial \dot{x}}{\partial q_1} = \frac{\partial^2 x}{\partial q_1 \partial t} + \frac{\partial^2 x}{\partial q_1^2}\dot{q}_1 + \frac{\partial^2 x}{\partial q_1 \partial q_2}\dot{q}_2 + \cdots + \frac{\partial^2 x}{\partial q_1 \partial q_n}\dot{q}_n,$$

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial \dot{x}}{\partial q_1}. \tag{2}$$

$$\delta_{q_1} W = \Sigma m \left[\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right]\delta q_1.$$

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1} \text{ by (1) and (2).}$$

$$\ddot{x}\frac{\partial x}{\partial q_1} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_1}\left(\frac{\dot{x}^2}{2}\right)\right] - \frac{\partial}{\partial q_1}\left(\frac{\dot{x}^2}{2}\right),$$

and therefore $$\delta_{q_1} W = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1}\right]\delta q_1,$$

and if $Q_1 \delta q_1$ is the work done by the actual forces when q_1 is changed by δq_1,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1. \tag{3}$$

If the geometrical equations do not contain the time, the same result is seen to hold, and in this case it is to be noted that since $\dot{x}$ is homogeneous of the first degree in the time derivatives of the coördinates, that is, in $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, the kinetic energy T is a homogeneous quadratic in $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$.

Generally every one of the n equations of which equation (3) is the type will contain all the n coördinates $q_1, q_2, \cdots, q_n$, and their time derivatives, and can be solved only by aid of the others. That is, we shall have n coördinates and the time connected by n simultaneous differential equations no one of which can ordinarily be solved by itself.

If the forces exerted by the connections and constraints do no work, they will not appear in our differential equations. Should we care to investigate any of them, we have only to suppose the constraints in question removed and the number of degrees of freedom correspondingly increased, and then to replace the constraints by the forces they exert and to form the full set of equations on the new hypothesis.

8. A System of Particles. Illustrative Examples. (*a*) A rough plank 16 feet long rests pointing downward on a smooth plane inclined at an angle of 30° to the horizon. A dog weighing as much as the plank runs down the plank just fast enough to keep it from slipping. What is his velocity when he reaches its lower end?

Here we have two degrees of freedom. Take x, the distance of the upper end of the plank from a fixed horizontal line in the plane, and y, the distance of the dog from the upper end of the plank, as coördinates, and let R be the backward force exerted by the dog on the plank, and m the weight of the dog.

$$T = \frac{m}{2}[\dot{x}^2 + (\dot{x}+\dot{y})^2] = \frac{m}{2}[2\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}].$$

$$\frac{\partial T}{\partial \dot{x}} = m[2\dot{x} + \dot{y}],$$

$$\frac{\partial T}{\partial \dot{y}} = m[\dot{x} + \dot{y}].$$

$$m[2\ddot{x} + \ddot{y}]\,\delta x = 2\,mg \sin 30^\circ\,\delta x.$$

$$m[\ddot{x} + \ddot{y}]\,\delta y = [R + mg\sin 30^\circ]\,\delta y.$$

$$2\ddot{x} + \ddot{y} = g,$$

$$\ddot{x} + \ddot{y} = \frac{R}{m} + \frac{g}{2}.$$

By hypothesis, $\quad x = \text{a constant},$

and therefore $\quad \ddot{y} = g.$

$$2\ddot{y}\dot{y} = 2g\dot{y},$$

$$\dot{y}^2 = 2gy + C = 2gy,$$

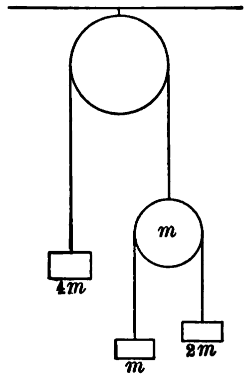
$$\dot{y} = \sqrt{2gy}.$$

When $y = 16$, $\quad \dot{y} = 32$, nearly.

Since $\quad \ddot{y} = \dfrac{R}{m} + \dfrac{g}{2},$

$$R = \frac{mg}{2}.$$

(*b*) A weight $4m$ is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to a smooth pulley of weight m, over which passes a second string attached to weights m and $2m$. The system starts from rest. Find the motion of the weight $4m$.



Two coördinates, x, the distance of $4m$ below the center of the fixed pulley, and y, the distance of $2m$ below the center of the movable pulley, will suffice. The velocities are

$$\dot{x} \text{ for } 4m,$$

$$-\dot{x} \text{ for movable pulley},$$

$$-\dot{x}+\dot{y} \text{ for } 2m,$$

$$-\dot{x}-\dot{y} \text{ for } m.$$

$$T = \tfrac{1}{2}[4m\dot{x}^2 + m\dot{x}^2 + 2m(\dot{y}-\dot{x})^2 + m(\dot{x}+\dot{y})^2]$$

$$= \frac{m}{2}[8\dot{x}^2 + 3\dot{y}^2 - 2\dot{x}\dot{y}].$$

$$\frac{\partial T}{\partial \dot{x}} = m[8\dot{x} - \dot{y}],$$

$$\frac{\partial T}{\partial \dot{y}} = m[3\dot{y} - \dot{x}].$$

$$m[8\ddot{x} - \ddot{y}]\delta x = [4mg - mg - 2mg - mg]\delta x.$$

$$m[3\ddot{y} - \ddot{x}]\delta y = [2mg - mg]\delta y.$$

$$8\ddot{x} - \ddot{y} = 0.$$

$$3\ddot{y} - \ddot{x} = g.$$

$$23\ddot{x} = g.$$

$$\ddot{x} = \frac{g}{23}.$$

The weight $4m$ will descend with uniform acceleration equal to one twenty-third the acceleration of gravity.

(*c*) *The dumb-bell problem.* Two equal particles, each of mass m, connected by a weightless rigid bar of length a, are set

moving in any way on a smooth horizontal plane. Find the subsequent motion.

We have three degrees of freedom. Let x, y, be the rectangular coördinates of the middle of the bar, and θ the angle made by the bar with the axis of X.

The rectangular coördinates of the two particles are

$$\left(x-\frac{a}{2}\cos\theta,\ y-\frac{a}{2}\sin\theta\right) \quad \text{and} \quad \left(x+\frac{a}{2}\cos\theta,\ y+\frac{a}{2}\sin\theta\right);$$

their velocities are

$$\sqrt{\left[\left(\dot{x}+\frac{a}{2}\sin\theta\dot{\theta}\right)^2+\left(\dot{y}-\frac{a}{2}\cos\theta\dot{\theta}\right)^2\right]}$$

and

$$\sqrt{\left[\left(\dot{x}-\frac{a}{2}\sin\theta\dot{\theta}\right)^2+\left(\dot{y}+\frac{a}{2}\cos\theta\dot{\theta}\right)^2\right]}.$$

$$T=\frac{m}{2}\left[\dot{x}^2+\dot{y}^2+\frac{a^2}{4}\dot{\theta}^2+(a\sin\theta-a\cos\theta)\dot{\theta}(\dot{x}+\dot{y})\right.$$
$$\left.+\dot{x}^2+\dot{y}^2+\frac{a^2}{4}\dot{\theta}^2+(a\cos\theta-a\sin\theta)\dot{\theta}(\dot{x}+\dot{y})\right]$$
$$=m\left[\dot{x}^2+\dot{y}^2+\frac{a^2}{4}\dot{\theta}^2\right].$$

$$\frac{\partial T}{\partial\dot{x}}=2m\dot{x},$$

$$\frac{\partial T}{\partial\dot{y}}=2m\dot{y},$$

$$\frac{\partial T}{\partial\dot{\theta}}=\frac{ma^2}{2}\dot{\theta}.$$

$$2m\ddot{x}\delta x=0,$$

$$2m\ddot{y}\delta y=0,$$

$$\frac{ma^2}{2}\ddot{\theta}\delta\theta=0.$$

$$\ddot{x}=0,$$

$$\ddot{y}=0,$$

$$\ddot{\theta}=0.$$

Hence the middle of the bar describes a straight line with uniform velocity, and the bar rotates with uniform angular velocity about its moving middle point.

EXAMPLE

Two Alpine climbers are roped together. One slips over a precipice, dragging the other after him. Find their motion while falling.

Ans. Their center of gravity describes a parabola. The rope rotates with uniform angular velocity about their moving center of gravity.

(*d*) Two equal particles are connected by a string which passes through a hole in a smooth horizontal table. The first particle is set moving on the table, at right angles with the string, with velocity $\sqrt{ag}$ where a is the distance of the particle from the hole. The hanging particle is drawn a short distance downward and then released. Find approximately the subsequent motion of the suspended particle.

Let x be the distance of the second particle below its position of equilibrium at the time t, and θ the angle described about the hole in the time t by the first particle.

$$T = \frac{m}{2}[\dot{x}^2 + \dot{x}^2 + (a-x)^2\dot{\theta}^2].$$

$$\frac{\partial T}{\partial \dot{x}} = 2\,m\dot{x},$$

$$\frac{\partial T}{\partial x} = -\,m\,(a-x)\dot{\theta}^2,$$

$$\frac{\partial T}{\partial \dot{\theta}} = m\,(a-x)^2\dot{\theta}.$$

$$m\,[2\,\ddot{x} + (a-x)\,\dot{\theta}^2]\,\delta x = mg\delta x, \qquad (1)$$

$$m\frac{d}{dt}[(a-x)^2\dot{\theta}]\,\delta\theta = 0. \qquad (2)$$

$$2\,\ddot{x} + (a-x)\,\dot{\theta}^2 = g. \qquad (3)$$

$$(a-x)^2\dot{\theta} = C = a\sqrt{ag}, \qquad (4)$$

since (2) holds good while the hanging particle is being drawn down as well as after it has been released.

$$2\ddot{x} + \frac{a^3 g}{(a-x)^3} = g,$$

$$2\ddot{x} + \left[\left(1 - \frac{x}{a}\right)^{-3} - 1\right] g = 0,$$

$$2\ddot{x} + \frac{3x}{a} g = 0, \text{ approximately,}$$

and

$$\ddot{x} + \frac{3g}{2a} x = 0.$$

For small oscillations of a simple pendulum of length l,

$$\ddot{\theta} + \frac{g}{l}\theta = 0.$$

Therefore the suspended particle will execute small oscillations, the length of the equivalent simple pendulum being $\frac{2}{3}a$.

EXAMPLE

A golf ball weighing one ounce and attached to a strong string is "teed up" on a large, smooth, horizontal table. The string is passed through a hole in the table, 10 feet from the ball, and fastened to a hundred-pound weight which rests on a prop just below the hole. The ball is then driven horizontally, at right angles with the string, with an initial velocity of a hundred feet a second, and the prop on which the weight rests is knocked away.

(*a*) How high must the table be to prevent the weight from falling to the ground? (*b*) What is the greatest velocity the golf ball will acquire? *Ans.* (*a*) 8.96 ft. (*b*) 963.4 ft. per sec.

9. Rigid Bodies. Two-dimensional Motion. If the particles of a system are so connected that they form a *rigid body* or a system of rigid bodies, the reasoning and formulas of Art. 7 still hold good.

(*a*) Let any rigid body containing a horizontal axis fixed in the body and fixed in space move under gravity. Suppose that the body cannot slide along the axis. Then the motion is obviously rotational, and there is but one degree of freedom. Take as the single coördinate the angle θ made by a plane containing the axis and the center of gravity of the body with a vertical plane through the axis.

Let h be the distance of the center of gravity from the axis, and k the radius of gyration of the body about a horizontal axis through the center of gravity. Then

$$T = \frac{m}{2}(h^2 + k^2)\dot{\theta}^2. \quad \text{(v. App. A, §§ 5 and 10)}$$

$$\frac{\partial T}{\partial \dot{\theta}} = m(h^2 + k^2)\dot{\theta}.$$

$$m(h^2 + k^2)\ddot{\theta}\delta\theta = -mgh\sin\theta\delta\theta.$$

$$\ddot{\theta} + \frac{gh}{h^2 + k^2}\sin\theta = 0,$$

and we have simple pendulum motion, the length of the equivalent simple pendulum being

$$l = \frac{h^2 + k^2}{h}.$$

(*b*) Two equal straight rods are connected by two equal strings of length a fastened to the ends of the rods, the whole forming a quadrilateral which is then suspended from a horizontal axis through the middle of the upper rod. The system is set moving in a vertical plane. Find the motion.

Take as coördinates ϕ, the inclination of the upper rod to the horizon, and θ, the angle made with the vertical by a line joining the point of suspension with the middle of the lower rod. From the nature of the connection the rods are always parallel.

Let k be the radius of gyration of each rod about its center of gravity.

$$T = \frac{m}{2}k^2\dot{\phi}^2 + \frac{m}{2}[a^2\dot{\theta}^2 + k^2\dot{\phi}^2] = \frac{m}{2}[2k^2\dot{\phi}^2 + a^2\dot{\theta}^2].$$

(v. App. A, § 10)

$$\frac{\partial T}{\partial \dot{\phi}} = 2mk^2\dot{\phi},$$

$$\frac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta}.$$

$$2mk^2\ddot{\phi}\delta\phi = 0,$$

$$\ddot{\phi} = 0.$$

$$\ddot{\theta} + \frac{g}{a}\sin\theta = 0,$$

and the rods revolve with uniform angular velocity while the middle point of the lower rod is oscillating as if it were the bob of a simple pendulum of length a.

(*c*) If an inclined plane is just rough enough to insure the rolling of a homogeneous cylinder, show that a thin hollow drum will roll and slip, the rate of slipping at any instant being one half the linear velocity.

Let x be the distance the axis of the cylinder has moved down the incline, θ the angle through which the cylinder has rotated, a the radius of the cylinder, and α the inclination of the plane. Call the force of friction F.

$$T = \frac{m}{2}[\dot{x}^2 + k^2\dot{\theta}^2].$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x},$$

$$\frac{\partial T}{\partial \dot{\theta}} = mk^2\dot{\theta}.$$

$$m\ddot{x}\delta x = [mg\sin\alpha - F]\,\delta x,$$

$$mk^2\ddot{\theta}\delta\theta = Fa\delta\theta.$$

If there is no slipping, $x = a\theta$,

$$m\ddot{x} = mg\sin\alpha - F,$$

$$mk^2\frac{\ddot{x}}{a} = Fa.$$

Hence $$\frac{Fa^2}{k^2} = mg \sin \alpha - F,$$

$$F = \frac{k^2}{a^2 + k^2} mg \sin \alpha.$$

For a solid cylinder, $$k^2 = \frac{a^2}{2},$$

$$F = \tfrac{1}{3} mg \sin \alpha;$$

$$R = mg \cos \alpha,$$

where R is the pressure on the plane;

$$\mu = \frac{F}{R} = \frac{1}{3} \tan \alpha,$$

where μ is the coefficient of friction.

For a hollow drum, $$k = a,$$

$$F = \tfrac{1}{2} mg \sin \alpha,$$

$$\frac{F}{R} = \frac{1}{2} \tan \alpha,$$

$$\mu < \tfrac{1}{2} \tan \alpha,$$

and the drum will slip.

For the drum, then,

$$F = \mu R = \mu mg \cos \alpha = \tfrac{1}{3} mg \sin \alpha,$$

$$m\ddot{x} = mg \sin \alpha - \tfrac{1}{3} mg \sin \alpha = \tfrac{2}{3} mg \sin \alpha,$$

$$\dot{x} = \tfrac{2}{3} gt \sin \alpha.$$

$$ma^2\ddot{\theta} = Fa = \frac{a}{3} mg \sin \alpha,$$

$$a\ddot{\theta} = \tfrac{1}{3} g \sin \alpha,$$

$$a\dot{\theta} = \tfrac{1}{3} gt \sin \alpha.$$

$$S = \dot{x} - a\dot{\theta} = \frac{1}{3} mgt \sin \alpha = \frac{\dot{x}}{2},$$

where S is the rate of slipping.

EXAMPLES

1. A sphere rotating about a horizontal axis is placed on a perfectly rough horizontal plane and rolls along in a straight line. Show that after the start friction exerts no force.

2. A sphere starting from rest moves down a rough inclined plane. Find the motion. (*a*) What must the coefficient of friction be to prevent slipping? (*b*) If there is slipping, what is its velocity?

Ans. (*a*) $\mu > \frac{2}{7}\tan\alpha$. (*b*) $S = gt[\sin\alpha - \frac{7}{2}\mu\cos\alpha]$.

3. A wedge of mass M having a smooth face and a perfectly rough face, making with each other an angle α, is placed with its smooth face on a horizontal table, and a sphere of mass m and radius a is placed on the wedge and rolls down. Find the motion.

Let x be the distance the wedge moves on the table, and y the distance the sphere rolls down the plane.

Note that $T = \dfrac{1}{2}[M+m]\dot{x}^2 + \dfrac{m}{2}\left[\dfrac{a^2+k^2}{a^2}\dot{y}^2 - 2\dot{x}\dot{y}\cos\alpha\right]$.

Ans. $(m+M)x - my\cos\alpha = 0,\quad \frac{7}{5}y - x\cos\alpha = \frac{1}{2}gt^2\sin\alpha$.

10. Rigid Bodies. Three-dimensional Motion. (*a*) A homogeneous sphere is set rolling in any way on a perfectly rough horizontal plane. Find the subsequent motion.

Let x, y, a, be the coördinates of the center of the sphere referred to a set of rectangular axes fixed in space; two of which, the axes of X and Y, lie in the given plane. Let OA, OB, OC, be rectangular axes fixed in the sphere and passing through its center; let OX, OY, OZ, be rectangular axes through the center of the sphere parallel to the axes fixed in space; and let θ, ϕ, ψ be the Euler's angles (v. App. A, § 8). Take x, y, θ, ϕ, and ψ as our coördinates. The only force we have to consider is F, the friction, and we shall let F_x and F_y be its components parallel to the axes OX, OY, respectively.

$$T = \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)],$$

where
$$\omega_x = -\dot\theta \sin\psi + \dot\phi \sin\theta \cos\psi,$$
$$\omega_y = \dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi,$$
$$\omega_z = \dot\phi \cos\theta + \dot\psi. \qquad \text{(v. App. A, § 8)}$$

Hence
$$T = \frac{m}{2}\left[\dot x^2 + \dot y^2 + k^2(\dot\theta^2 + \dot\phi^2 + \dot\psi^2 + 2\cos\theta\dot\phi\dot\psi)\right].$$

We get
$$m\ddot x = F_x, \tag{1}$$
$$m\ddot y = F_y, \tag{2}$$
$$mk^2 \frac{d}{dt}[\dot\psi + \cos\theta\dot\phi] = 0, \tag{3}$$
$$mk^2 \frac{d}{dt}[\dot\phi + \cos\theta\dot\psi] = -aF_x \sin\theta\sin\psi + aF_y \sin\theta\cos\psi, \tag{4}$$
$$mk^2[\ddot\theta + \sin\theta\dot\phi\dot\psi] = -aF_x\cos\psi - aF_y\sin\psi; \tag{5}$$

and as there is no slipping,
$$\dot x - a\omega_y = \dot x - a(\dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi) = 0, \tag{6}$$
$$\dot y + a\omega_x = \dot y + a(-\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi) = 0. \tag{7}$$

From (4) and (5),
$$mk^2\left[\sin\psi\frac{d}{dt}(\dot\phi + \cos\theta\dot\psi) + \sin\theta\cos\psi(\ddot\theta + \sin\theta\dot\phi\dot\psi)\right] = -aF_x\sin\theta, \tag{8}$$
$$mk^2\left[\cos\psi\frac{d}{dt}(\dot\phi + \cos\theta\dot\psi) - \sin\theta\sin\psi(\ddot\theta + \sin\theta\dot\phi\dot\psi)\right] = aF_y\sin\theta. \tag{9}$$

Expanding the first members of (8) and (9) and eliminating $\ddot\psi$ by the aid of (3), we get
$$mk^2[\cos\psi\ddot\theta - \sin\psi\dot\theta\dot\psi + \sin\theta\sin\psi\ddot\phi + \cos\theta\sin\psi\dot\theta\dot\phi + \sin\theta\cos\psi\dot\phi\dot\psi] = -aF_x, \tag{10}$$
$$mk^2[-\sin\psi\ddot\theta - \cos\psi\dot\theta\dot\psi + \sin\theta\cos\psi\ddot\phi + \cos\theta\cos\psi\dot\theta\dot\phi - \sin\theta\sin\psi\dot\phi\dot\psi] = aF_y. \tag{11}$$

But the first members of (10) and (11) are obviously $mk^2\dfrac{d\omega_y}{dt}$ and $mk^2\dfrac{d\omega_x}{dt}$, respectively. Hence, by (6) and (7),

$$\frac{mk^2}{a}\ddot{x} = -aF_x,$$

$$-\frac{mk^2}{a}\ddot{y} = aF_y.$$

Substituting in (1) and (2), we get

$$\frac{mk^2}{a}\ddot{x} = -ma\ddot{x},$$

$$\frac{mk^2}{a}\ddot{y} = -ma\ddot{y};$$

whence
$$\ddot{x} = 0,$$
$$\ddot{y} = 0.$$

From (7) and (6),
$$\frac{d\omega_x}{dt} = 0,$$
$$\frac{d\omega_y}{dt} = 0;$$

and from (3),
$$\frac{d\omega_z}{dt} = 0.$$

Finally,
$$F_x = 0,$$
$$F_y = 0.$$

Hence the center of the sphere moves in a straight line with uniform velocity, and the sphere rotates with uniform angular velocity about an instantaneous axis which does not change its direction, and no friction is brought into play after the rolling begins.

(*b*) *The billiard ball.* Suppose the horizontal table in (*a*) is imperfectly rough, coefficient of friction μ, and suppose the ball to slip.

Take the same coördinates as before, and equations (1), (2), (3), (4), (5), (8), (9), (10), and (11) still hold good. Let α

be the angle the direction of the resultant friction, $F = \mu mg$, makes with the axis of X, and let S be the velocity of slipping, that is, the velocity with which the lowest point of the ball moves along the table. Of course the directions of F and S are opposite.

Let S_x and S_y be the components of S parallel to the axes of X and Y. We have

$$-S\cos\alpha = S_x = \dot{x} - a\omega_y,$$

and

$$-S\sin\alpha = S_y = \dot{y} + a\omega_x.$$

$$F_x = \mu mg\cos\alpha,$$

and

$$F_y = \mu mg\sin\alpha.$$

$$\frac{dS_x}{dt} = S\sin\alpha\frac{d\alpha}{dt} - \cos\alpha\frac{dS}{dt} = \ddot{x} - a\frac{d\omega_y}{dt}.$$

$$\frac{dS_y}{dt} = -S\cos\alpha\frac{d\alpha}{dt} - \sin\alpha\frac{dS}{dt} = \ddot{y} + a\frac{d\omega_x}{dt}.$$

From (1), $$\ddot{x} = \mu g\cos\alpha,$$

and from (10), $$\frac{d\omega_y}{dt} = -\frac{a}{k^2}\mu g\cos\alpha.$$

Hence $$S\sin\alpha\frac{d\alpha}{dt} - \cos\alpha\frac{dS}{dt} = \frac{a^2+k^2}{k^2}\mu g\cos\alpha, \tag{12}$$

and from (2) and (11),

$$-S\cos\alpha\frac{d\alpha}{dt} - \sin\alpha\frac{dS}{dt} = \frac{a^2+k^2}{k^2}\mu g\sin\alpha. \tag{13}$$

Multiplying (12) by $\sin\alpha$ and (13) by $\cos\alpha$, and subtracting,

$$S\frac{d\alpha}{dt} = 0. \tag{14}$$

Multiplying (12) by $\cos\alpha$ and (13) by $\sin\alpha$, and adding,

$$-\frac{dS}{dt} = \frac{a^2+k^2}{k^2}\mu g. \tag{15}$$

Integrating (15), $S = S_0 - \frac{a^2 + k^2}{k^2} \mu g t.$ (16)

From (14), $\alpha = \alpha_0,$

and the direction of slipping does not change.

If the axes are so chosen that the axis of X has the direction opposite to the direction of slipping, $\alpha = 0$. Then

$$\ddot{x} = \mu g,$$
$$\ddot{y} = 0.$$

These equations are familiar in the theory of projectiles, and the path traced on the table is a parabola so long as slipping lasts.

Should $\dot{y}_0$ happen to be zero, the path degenerates into a straight line.

When slipping stops,

$$\dot{x} - a\omega_y = 0, \quad \text{and} \quad \dot{y} + a\omega_x = 0,$$

and we have the case treated in (*a*).

EXAMPLE

A homogeneous sphere is set rolling on a perfectly rough inclined plane. Find the path traced on the plane.

Ans. A parabola.

(*c*) *The gyroscope.* Suppose a rigid body containing a fixed point and having two of its moments of inertia about its principal axes through the fixed point equal. Obtain the differential equations for its motion under gravity.

We shall use Euler's angles with a vertical axis of Z.

We have
$$\omega_1 = \dot{\theta} \sin\phi - \dot{\psi} \sin\theta \cos\phi,$$
$$\omega_2 = \dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi,$$
$$\omega_3 = \dot{\psi} \cos\theta + \dot{\phi}. \qquad \text{(v. App. A, § 8)}$$
$$T = \tfrac{1}{2}\left[A\omega_1^2 + A\omega_2^2 + C\omega_3^2\right] \qquad \text{(v. App. A, § 10)}$$
$$= \tfrac{1}{2}\left[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + C(\dot{\psi}\cos\theta + \dot{\phi})^2\right].$$

$$\frac{\partial T}{\partial \dot\phi} = C(\dot\psi \cos\theta + \dot\phi),$$

$$\frac{\partial T}{\partial \dot\psi} = A \sin^2\theta\dot\psi + C\cos\theta(\dot\psi\cos\theta + \dot\phi),$$

$$\frac{\partial T}{\partial \dot\theta} = A\dot\theta,$$

$$\frac{\partial T}{\partial \theta} = A\sin\theta\cos\theta\dot\psi^2 - C\sin\theta(\dot\psi\cos\theta + \dot\phi)\dot\psi.$$

Our equations are $$C\frac{d}{dt}(\dot\psi\cos\theta + \dot\phi) = 0, \tag{1}$$

$$\frac{d}{dt}[A\sin^2\theta\dot\psi + C\cos\theta(\dot\psi\cos\theta + \dot\phi)] = 0, \tag{2}$$

$$A\ddot\theta - A\sin\theta\cos\theta\dot\psi^2 + C\sin\theta(\dot\psi\cos\theta + \dot\phi)\dot\psi = mga\sin\theta. \tag{3}$$

From (1), $$\dot\psi\cos\theta + \dot\phi = \alpha, \tag{4}$$

where α is the initial velocity about the axis of unequal moment.

$$A\sin^2\theta\dot\psi + C\alpha\cos\theta = L. \tag{5}$$

$$A\ddot\theta - A\sin\theta\cos\theta\dot\psi^2 + C\alpha\sin\theta\dot\psi = mga\sin\theta, \tag{6}$$

or substituting $\dot\psi$ from (5),

$$A\ddot\theta = \frac{(L - C\alpha\cos\theta)(L\cos\theta - C\alpha)}{A\sin^3\theta} + mga\sin\theta. \tag{7}$$

(*d*) Obtain Euler's equations for a rigid body containing a fixed point.

Here $$T = \tfrac{1}{2}[A\omega_1^2 + B\omega_2^2 + C\omega_3^2]. \qquad \text{(v. App. A, § 10)}$$

$$\frac{\partial T}{\partial \dot\phi} = C\omega_3, \qquad \text{(v. App. A, § 8)}$$

$$\frac{\partial T}{\partial \phi} = A\omega_1[\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi] + B\omega_2[-\dot\theta\sin\phi + \dot\psi\sin\theta\cos\phi]$$
$$= A\omega_1\omega_2 - B\omega_2\omega_1.$$

Whence $$\left[C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2\right]\delta\phi=N\delta\phi,$$

where N is the moment of the impressed forces about the C axis.

$$C\frac{d\omega_3}{dt}-(A-B)\,\omega_1\omega_2=N.$$

The remaining two Euler's equations follow at once from this by considerations of symmetry.

11. In Arts. 2 and 7 it was shown that under slight limitations the coördinates of a moving particle or of a moving system could be taken practically at pleasure, and the differential equations of motion could be obtained by the application of a single formula. It does not follow, however, that when it comes to solving a concrete problem completely, the choice of coördinates is a matter of indifference. Different possible choices may lead to differential equations differing greatly in complication, and as a matter of fact in the illustrative problems of the present chapter the coördinates have been selected with care and judgment. That this care, while convenient, is not essential may be worth illustrating by a practical example, and we shall consider the simple familiar case of a projectile *in vacuo*.

Altogether the simplest coördinates are x and y, rectangular coördinates referred to a horizontal axis of X and a vertical axis of Y through the point of projection.

We have $$T=\frac{m}{2}(\dot{x}^2+\dot{y}^2),$$

$$\frac{\partial T}{\partial \dot{x}}=m\dot{x},$$

$$\frac{\partial T}{\partial \dot{y}}=m\dot{y}.$$

$$m\ddot{x}=0,$$

$$m\ddot{y}=-mg.$$

Solving,

$$\dot{x} = v_x,$$

$$\dot{y} = v_y - gt.$$

$$x = v_x t,$$

$$y = v_y t + \frac{gt^2}{2}.$$

Let us now try a perfectly crazy set of coördinates, q_1 and q_2, where

$$q_1 = x + \tan^{-1} y,$$

and

$$q_2 = x - \tan^{-1} y.$$

Proceeding in our regular way, we have

$$x = \tfrac{1}{2}(q_1 + q_2),$$

$$y = \tan \frac{q_1 - q_2}{2}.$$

$$\dot{x} = \tfrac{1}{2}(\dot{q}_1 + \dot{q}_2),$$

$$\dot{y} = \frac{1}{2} \sec^2 \frac{q_1 - q_2}{2} (\dot{q}_1 - \dot{q}_2).$$

$$T = \frac{m}{8}\left[\dot{q}_1^2 + 2\,\dot{q}_1\dot{q}_2 + \dot{q}_2^2 + \sec^4 \frac{q_1 - q_2}{2} (\dot{q}_1^2 - 2\,\dot{q}_1\dot{q}_2 + \dot{q}_2^2)\right].$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{m}{4}\left[\dot{q}_1 + \dot{q}_2 + \sec^4 \frac{q_1 - q_2}{2} (\dot{q}_1 - \dot{q}_2)\right],$$

$$\frac{\partial T}{\partial q_1} = \frac{m}{4} \sec^4 \frac{q_1 - q_2}{2} \tan \frac{q_1 - q_2}{2} (\dot{q}_1 - \dot{q}_2)^2,$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{m}{4}\left[\dot{q}_1 + \dot{q}_2 - \sec^4 \frac{q_1 - q_2}{2} (\dot{q}_1 - \dot{q}_2)\right],$$

$$\frac{\partial T}{\partial q_2} = -\frac{m}{4} \sec^4 \frac{q_1 - q_2}{2} \tan \frac{q_1 - q_2}{2} (\dot{q}_1 - \dot{q}_2)^2.$$

$$\frac{m}{4}\left[\ddot{q}_1 + \ddot{q}_2 + \sec^4 \frac{q_1 - q_2}{2} (\ddot{q}_1 - \ddot{q}_2) + \sec^4 \frac{q_1 - q_2}{2} \tan \frac{q_1 - q_2}{2} (\dot{q}_1 - \dot{q}_2)^2\right]$$

$$= -\frac{mg}{2} \sec^2 \frac{q_1 - q_2}{2}. \quad (1)$$

$$\frac{m}{4}\left[\ddot{q}_1+\ddot{q}_2-\sec^4\frac{q_1-q_2}{2}(\ddot{q}_1-\ddot{q}_2)-\sec^4\frac{q_1-q_2}{2}\tan\frac{q_1-q_2}{2}(\dot{q}_1-\dot{q}_2)^2\right]$$
$$=\frac{mg}{2}\sec^2\frac{q_1-q_2}{2}. \qquad (2)$$

Adding (1) and (2), $\frac{m}{2}(\ddot{q}_1+\ddot{q}_2)=0.$

Whence $$\dot{q}_1+\dot{q}_2=2v_x,$$

and $$q_1+q_2=2v_xt. \qquad (3)$$

Subtracting (2) from (1),

$$\frac{m}{2}\left[\sec^4\frac{q_1-q_2}{2}(\ddot{q}_1-\ddot{q}_2)+\sec^4\frac{q_1-q_2}{2}\tan\frac{q_1-q_2}{2}(\dot{q}_1-\dot{q}_2)^2\right]$$
$$=-mg\sec^2\frac{q_1-q_2}{2}.$$

Multiplying by $\frac{4}{m}(\dot{q}_1-\dot{q}_2)$, and integrating,

$$\sec^4\frac{q_1-q_2}{2}(\dot{q}_1-\dot{q}_2)^2=-8\tan\frac{q_1-q_2}{2}+4v_y^2.$$

$$\sec^2\frac{q_1-q_2}{2}(\dot{q}_1-\dot{q}_2)=2\sqrt{v_y^2-2g\tan\frac{q_1-q_2}{2}}.$$

Let $$z=\frac{q_1-q_2}{2}.$$

$$\sec^2 z\frac{dz}{dt}=\sqrt{v_y^2-2g\tan z},$$

$$dt=\frac{\sec^2 z\,dz}{\sqrt{v_y^2-2g\tan z}},$$

$$t=\frac{1}{g}\left[v_y-\sqrt{y_y^2-2g\tan z}\right]$$

$$=\frac{1}{g}\left[v_y-\sqrt{v_y^2-2g\tan\frac{q_1-q_2}{2}}\right].$$

$$(v_y-gt)^2=v_y^2-2g\tan\frac{q_1-q_2}{2},$$

$$\tan\frac{q_1-q_2}{2}=v_yt-\frac{gt^2}{2},$$

$$\frac{q_1-q_2}{2}=\tan^{-1}\left[v_yt-\frac{gt^2}{2}\right].$$

But from (3), $\frac{q_1 + q_2}{2} = v_x t$,

Hence
$$q_1 = v_x t + \tan^{-1}\left[v_y t - \frac{gt^2}{2}\right],$$
$$q_2 = v_x t - \tan^{-1}\left[v_y t - \frac{gt^2}{2}\right].$$

Of course this should agree with our first answer, and a moment's consideration shows that it does.

We have
$$\frac{q_1 + q_2}{2} = x = v_x t,$$
$$\tan\frac{q_1 - q_2}{2} = y = v_y t - \frac{gt^2}{2}, \text{ as before.}$$

12. The parameters $q_1, q_2, \cdots$ that we have been using to fix the position of our moving particle or moving system are called *generalized coördinates.* Following the analogy of rectangular coördinates, the time derivative $\dot{q}_k$ of any generalized coördinate q_k is called the generalized component of velocity corresponding to q_k. It may be a linear velocity, or an angular velocity as in many of our problems, or it may be much more complicated than either as in our latest example.

The kinetic energy T expressed in terms of the generalized coördinates and the generalized velocities is called the *Lagrangian expression* for the kinetic energy.

If we are using rectangular coördinates and dealing with a moving particle,
$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2);$$
$$\frac{\partial T}{\partial \dot{x}} = m\dot{x},$$
and is the x component of the momentum of the particle.

Following this analogy, $\frac{\partial T}{\partial \dot{q}_k}$ is called the generalized component of the *momentum* of the system, corresponding to the coördinate q_k. It is frequently represented by p_k, and may be a

momentum, or a moment of momentum as in many of our problems, or it may be much more complicated than either as in our latest example.

Equations of the type

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k$$

are practically what are called the *Lagrangian* equations of motion, although strictly speaking the regulation form of the Lagrangian equations is a little more compact and will be given later, in Chapter IV.

Q_k, defined through the property that $Q_k \delta q_k$ is the work done by the actual forces when q_k is changed by δq_k, is called the generalized component of *force* corresponding to q_k. It may be a force, or the moment of a force as in many of our problems, or it may be much more complicated than either as in our latest example.

13. Summary of Chapter I. If a moving system has a finite number n of degrees of freedom (v. Art. 7) and n independent generalized coördinates $q_1, q_2, \cdots, q_n$, are chosen, the *kinetic energy* T can be expressed in terms of the coördinates and the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, and when so expressed will be a quadratic in the velocities, a homogeneous quadratic if the *geometrical equations* (v. Art. 7) do not contain the time explicitly.

The work done by the *effective forces* in a hypothetical infinitesimal displacement of the system due to an infinitesimal change ∂q_k in a single coördinate q_k is

$$\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}\right]\delta q_k.$$

If this is written equal to $Q_k \delta q_k$, the work done by the actual forces in the displacement in question, there will result the *Lagrangian* equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k.$$

The n equations of which this is the type form a set of simultaneous differential equations of the second order, connecting the n generalized coördinates with the time. When the *complete solution* of this set of equations has been obtained, the problem of the motion of the system is solved completely.

It must be kept in mind that in order to obtain the value of a single coördinate or of a set of coördinates less in number than n it is generally necessary to form and to solve the complete set of n differential equations.

We shall see, however, in the next chapter, that in certain important classes of problems some of these equations need not be formed, and that some of the coördinates can be safely ignored without interfering with our obtaining the values of the remaining coördinates; that, indeed, we may be able to handle satisfactorily some problems concerning moving systems having an infinite number of degrees of freedom.

CHAPTER II

THE HAMILTONIAN EQUATIONS. ROUTH'S MODIFIED LAGRANGIAN EXPRESSION. IGNORATION OF COÖRDINATES

14. The Hamiltonian Equations. If the geometrical equations of the system (v. Art. 7) do not contain the time explicitly and the *kinetic energy* T is therefore a homogeneous quadratic in $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, the generalized component velocities, *Lagrange's equations* can be replaced by a set known as the *Hamiltonian equations.*

The Lagrangian expression for the kinetic energy we shall now represent by $T_{\dot{q}}$.

Let $p_1 = \dfrac{\partial T_{\dot{q}}}{\partial \dot{q}_1}$, $p_2 = \dfrac{\partial T_{\dot{q}}}{\partial \dot{q}_2}$, etc. be the generalized component momenta. Then $p_1, p_2, \cdots$ are homogeneous of the first degree in $\dot{q}_1, \dot{q}_2, \cdots$. Express $\dot{q}_1, \dot{q}_2, \cdots$ in terms of $p_1, p_2, \cdots, q_1, q_2, \cdots$, noting that they are homogeneous of the first degree in terms of $p_1, p_2, \cdots$, and substitute these values for them in $T_{\dot{q}}$, which will thus become an explicit function of the momenta and the coördinates, homogeneous of the second degree in terms of the former. This function is called the *Hamiltonian expression* for the kinetic energy, and we shall represent it by T_p. Of course

$$T_{\dot{q}} \equiv T_p. \tag{1}$$

By Euler's Theorem,

$$2\,T_{\dot{q}} = \dot{q}_1 \frac{\partial T_{\dot{q}}}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T_{\dot{q}}}{\partial \dot{q}_2} + \cdots,$$

therefore
$$2\,T_{\dot{q}} \equiv 2\,T_p = p_1\dot{q}_1 + p_2\dot{q}_2 + \cdots. \tag{2}$$

Let us try to get $\dfrac{\partial T_p}{\partial q_1}$ and $\dfrac{\partial T_p}{\partial p_1}$ indirectly.

From (1), $$\frac{\partial T_p}{\partial q_1} = \frac{\partial T_{\dot{q}}}{\partial q_1} + \frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial T_{\dot{q}}}{\partial \dot{q}_2}\frac{\partial \dot{q}_2}{\partial q_1} + \cdots,$$

or $$\frac{\partial T_p}{\partial q_1} = \frac{\partial T_{\dot{q}}}{\partial q_1} + p_1\frac{\partial \dot{q}_1}{\partial q_1} + p^2\frac{\partial \dot{q}_2}{\partial q_1} + \cdots. \tag{3}$$

But from (2),

$$2\frac{\partial T_p}{\partial q_1} = p_1\frac{\partial \dot{q}_1}{\partial q_1} + p_2\frac{\partial \dot{q}_2}{\partial q_1} + \cdots. \tag{4}$$

Subtracting (3) from (4), we get

$$\frac{\partial T_p}{\partial q_1} = -\frac{\partial T_{\dot{q}}}{\partial q_1}. \tag{5}$$

Again, we have from (1),

$$\frac{\partial T_p}{\partial p_1} = \frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial p_1} + \frac{\partial T_{\dot{q}}}{\partial \dot{q}_2}\frac{\partial \dot{q}_2}{\partial p_1} + \cdots,$$

or $$\frac{\partial T}{\partial p_1} = p_1\frac{\partial \dot{q}_1}{\partial p_1} + p_2\frac{\partial \dot{q}_2}{\partial p_1} + \cdots. \tag{6}$$

From (2), $$2\frac{\partial T_p}{\partial p_1} = \dot{q}_1 + p_1\frac{\partial \dot{q}_1}{\partial p_1} + p_2\frac{\partial \dot{q}_2}{\partial p_1} + \cdots. \tag{7}$$

Subtracting (6) from (7), we get

$$\frac{\partial T_p}{\partial p_1} = \dot{q}_1. \tag{8}$$

The Lagrangian equation

$$\frac{d}{dt}\frac{\partial T_{\dot{q}}}{\partial \dot{q}_k} - \frac{\partial T_{\dot{q}}}{\partial q_k} = Q_k$$

becomes $$\dot{p}_k + \frac{\partial T_p}{\partial q_k} = Q_k. \tag{9}$$

We have also $$\dot{q}_k = \frac{\partial T_p}{\partial p_k}. \tag{10}$$

The equations of which (9) and (10) are the type are known as the *Hamiltonian equations* of motion. The so-called *canonical form* of the Hamiltonian equations is somewhat more compact and will be given later, in Chapter IV.

The $2n$ equations of which (9) and (10) are the type form a system of $2n$ simultaneous differential equations of the first order, connecting the n coördinates $q_1, q_2, \cdots, q_n$, and the n component momenta $p_1, p_2, \cdots, p_n$, with the time, and in order to solve for any one coördinate we must generally, as in the case of the Lagrangian equations, form and make use of the whole set of equations.

In concrete problems there is usually no advantage in using the Hamiltonian forms, but in many theoretical investigations they are of importance. It may be noted that in the process of forming T_p from $T_{\dot q}$, $\dot q_1$ is expressed in terms of the p's and q's, and thus equation (10) is anticipated.

To familiarize the student with the actual working of the Hamiltonian forms, we shall apply them to a few problems which we have solved already by the Lagrange process.

15. (*a*) The equations of motion in a plane in terms of polar coördinates (v. Art. 3, (*a*)).

Here
$$T_{\dot q} = \frac{m}{2}[\dot r^2 + r^2\dot\phi^2].$$
$$\frac{\partial T_{\dot q}}{\partial \dot r} = p_r = m\dot r,$$
$$\frac{\partial T_{\dot q}}{\partial \dot\phi} = p_\phi = mr^2\dot\phi.$$

Whence
$$\dot r = \frac{p_r}{m}, \tag{1}$$
$$\dot\phi = \frac{p_\phi}{mr^2}, \tag{2}$$

and
$$T_p = \frac{1}{2m}\left[p_r^2 + \frac{p_\phi^2}{r^2}\right].$$
$$\frac{\partial T_p}{\partial r} = -\frac{p_\phi^2}{mr^3},$$
$$\frac{\partial T_p}{\partial \phi} = 0.$$

$$\left[\dot{p}_r - \frac{p_\phi^2}{mr^3}\right]\delta r = R\delta r, \tag{3}$$

$$\dot{p}_\phi \delta\phi = \Phi r \delta\phi. \tag{4}$$

Our Hamiltonian equations are

$$\dot{r} = \frac{p_r}{m}, \tag{5}$$

$$\dot{p}_r - \frac{p_\phi^2}{mr^3} = R, \tag{6}$$

$$\dot{\phi} = \frac{p_\phi}{mr^2}, \tag{7}$$

$$\dot{p}_\phi = r\Phi. \tag{8}$$

If we eliminate p_r and p_ϕ, we get

$$\frac{m}{r}\frac{d}{dt}(r^2\dot{\phi}) = \Phi,$$

$$m[\ddot{r} - r^2\dot{\phi}^2] = R,$$

our familiar equations.

(*b*) Motion of a bead on a horizontal circular wire (v. Art. 5, (*a*)).

Here
$$T_{\dot{q}} = \frac{m}{2}a^2\dot{\theta}^2.$$

$$\frac{\partial T_{\dot{q}}}{\partial\dot{\theta}} = p_\theta = ma^2\dot{\theta}.$$

$$\dot{\theta} = \frac{p_\theta}{ma^2}.$$

$$T_p = \frac{p_\theta^2}{2\,ma^2}.$$

$$\dot{p}_\theta\delta\theta = -\,ka^2\dot{\theta}^2 a\delta\theta = -\frac{kp_\theta^2}{m^2a}\delta\theta,$$

$$\frac{\dot{p}_\theta}{p_\theta^2} + \frac{k}{m^2a} = 0.$$

Integrating, $$-\frac{1}{p_\theta}+\frac{kt}{m^2a}=C=-\frac{1}{maV}.$$

$$p_\theta=\frac{m^2aV}{Vkt+m}.$$

$$\dot\theta=\frac{m}{ak}\frac{Vk}{Vkt+m}.$$

$$\theta=\frac{m}{ak}\log\left[Vkt+m\right]+C=\frac{m}{ak}\log\left[1+\frac{kVt}{m}\right].$$

(*c*) The tractrix problem (v. Art. 6, (*a*)).

Here $$T_{\dot q}=\frac{m}{2}\left[\dot x^2+a^2\dot\theta^2-2a\cos\theta\dot x\dot\theta\right].$$

$$p_x=m\left[\dot x-a\cos\theta\dot\theta\right],$$

$$p_\theta=m\left[a^2\dot\theta-a\cos\theta\dot x\right].$$

Whence $$\dot x=\frac{1}{ma\sin^2\theta}\left[p_x+\cos\theta p_\theta\right], \tag{1}$$

$$\dot\theta=\frac{1}{ma^2\sin^2\theta}\left[a\cos\theta p_x+p_\theta\right]. \tag{2}$$

$$T_p=\frac{1}{2\,ma^2\sin^2\theta}\left[a^2p_x^2+p_\theta^2+2\,a\cos\theta p_xp_\theta\right].$$

We get $$\dot p_x=R\sin\theta, \tag{3}$$

$$\dot p_\theta-\frac{1}{ma^2\sin^3\theta}\left[(a^2p_x^2+p_\theta^2)\cos\theta+a(1+\cos^2\theta)p_xp_\theta\right]=0. \tag{4}$$

We have the condition

$$x=nt. \tag{5}$$

With (5), $$p_x=m\left[n-a\cos\theta\dot\theta\right],$$

$$p_\theta=ma\left[a\dot\theta-n\cos\theta\right].$$

Substituting in (4), we get

$$\dot p_\theta-mna\sin\theta\dot\theta=0,$$

or $$ma^2\ddot\theta=0.$$

Whence $$\dot{\theta} = C = \frac{n}{a}.$$

$$p_x = mn(1 - \cos\theta).$$

$$\dot{p}_x = mn \sin\theta\dot{\theta} = \frac{mn^2}{a}\sin\theta = R\sin\theta.$$

$$R = \frac{mn^2}{a},$$

as in Art. 6.

(*d*) The problem of the two particles and the table with a hole in it (v. Art. 8, (*d*)).

Here $$T_{\dot{q}} = \frac{m}{2}[2\dot{x}^2 + (a-x)^2\dot{\theta}^2].$$

$$p_x = 2m\dot{x},$$

$$p_\theta = m(a-x)^2\dot{\theta}.$$

$$\dot{x} = \frac{p_x}{2m},$$

$$\dot{\theta} = \frac{p_\theta}{m(a-x)^2}.$$

$$T_p = \frac{1}{2m}\left[\frac{p_x^2}{2} + \frac{p_\theta^2}{(a-x)^2}\right].$$

We get $$\dot{p}_x + \frac{p_\theta^2}{m(a-x)^3} = mg,$$

and $$\dot{p}_\theta = 0.$$

$$p_\theta = C = ma\sqrt{ag}.$$

$$\dot{p}_x + \frac{ma^3}{(a-x)^3}g = mg.$$

Whence $$2\ddot{x} + \frac{3g}{a}x = 0$$

if x is small, as in Art. 8, (*d*).

(*e*) The gyroscope (v. Art. 10, (*c*)).

$$T_{\dot q} = \tfrac{1}{2}\left[A(\dot\theta^2 + \sin^2\theta\dot\psi^2) + C(\dot\psi\cos\theta + \dot\phi)^2\right].$$

$$p_\phi = \frac{\partial T_{\dot q}}{\partial\dot\phi} = C(\dot\psi\cos\theta + \dot\phi),$$

$$p_\psi = \frac{\partial T_{\dot q}}{\partial\dot\psi} = A\sin^2\theta\dot\psi + C\cos\theta(\dot\psi\cos\theta + \dot\phi),$$

$$p_\theta = \frac{\partial T_{\dot q}}{\partial\dot\theta} = A\dot\theta.$$

$$T_p = \frac{1}{2}\left\{\frac{1}{A}\left[p_\theta^2 + \frac{(p_\psi - p_\phi\cos\theta)^2}{\sin^2\theta}\right] + \frac{p_\phi^2}{C}\right\}.$$

We get
$$\dot p_\phi = 0, \tag{1}$$
$$\dot p_\psi = 0, \tag{2}$$
$$\dot p_\theta - \frac{1}{A\sin^3\theta}\left[(p_\phi^2 + p_\psi^2)\cos\theta - (1+\cos^2\theta)p_\phi p_\psi\right] = mga\sin\theta. \tag{3}$$

$$p_\phi = C\alpha,$$
$$p_\psi = L.$$

$$\dot p_\theta - \frac{1}{A\sin^3\theta}\left[(C^2\alpha^2 + L^2)\cos\theta - CL\alpha(1+\cos^2\theta)\right] = mga\sin\theta,$$

or
$$A\ddot\theta = \frac{(L - C\alpha\cos\theta)(L\cos\theta - C\alpha)}{A\sin^3\theta} + mga\sin\theta. \tag{4}$$

$$\dot\psi\cos\theta + \dot\phi = \alpha,$$
$$A\sin^2\theta\dot\psi + C\alpha\cos\theta = L,$$

as in Art. 10, (*c*).

16. The last two problems have a peculiarity that deserves closer examination. Let us consider Art. 15, (*e*). The kinetic energy in the Lagrangian form $T_{\dot q}$, and therefore* in the Hamiltonian form T_p, fails to contain the coördinates ϕ and ψ. Moreover, when either of these coördinates is varied, the

* Since $\dfrac{\partial T_p}{\partial q_k} = -\dfrac{\partial T_{\dot q}}{\partial q_k}$ (v. Art. 14), it follows that if a coördinate is missing in $T_{\dot q}$, it is missing also in T_p.

impressed forces do no work. Hence two of our Hamiltonian equations assume the very simple forms

$$\dot{p}_\phi = 0, \qquad \dot{p}_\psi = 0,$$

which give immediately

$$p_\phi = C\alpha, \text{ a constant,}$$

and $$p_\psi = L, \text{ a constant.}$$

These enable us to eliminate p_ϕ and p_ψ from a third Hamiltonian equation (Art. 15, (*e*), (3)), which then contains only the third coördinate θ and its corresponding momentum p_θ.

This same result might have been obtained just as well by replacing p_ϕ and p_ψ in T_p by their constant values and then forming the Hamiltonian equations for θ in the regular way. So that if we are interested in θ only, and T_p has once been formed and simplified by the substitution of constants for p_ϕ and p_ψ, the coördinates ϕ and ψ need play no further part in the solution. Should we care to get the values of these *ignored* coördinates, they can be found from the equations $p_\phi = C\alpha$, $p_\psi = L$, by the aid of the value of θ previously determined.

In Art. 15, (*d*), since $\dot{p}_\theta = 0$ and $p_\theta = ma\sqrt{ag}$, the substitution of this value for p_θ in T_p enables us to solve the problem so far as x is concerned without paying further attention to θ.

To generalize, it is easily seen that if the Lagrangian form, and therefore the Hamiltonian form, of the kinetic energy fails to contain some of the coördinates of a moving system,* and if the impressed forces are such that when any one of these coördinates is varied no work is done, the momenta $p_1, p_2, \cdots$ corresponding to these coördinates are constant; and that after substituting these constants for the momenta in question in the Hamiltonian form of the kinetic energy, the coördinates corresponding to them may be ignored in forming and in solving the Hamiltonian equations for the remaining coördinates.

* Coördinates that do not appear in the expression for the kinetic energy of a moving system are often called *cyclic* coördinates.

Unfortunately the ignored coördinates have to be used in forming $T_{\dot{q}}$, the Lagrangian form of the kinetic energy, and in deducing from it T_p, the Hamiltonian form of the energy.

Not infrequently this preliminary labor may be abridged considerably by using a modified form of the Lagrangian expression for the kinetic energy of the system, as we shall proceed to show.

17. Routh's Modified Form of the Lagrangian Expression for the Kinetic Energy of a Moving System. In forming the Hamiltonian equations of motion (v. Art. 14) we first changed the form of $T_{\dot{q}}$ by replacing all the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots$ by their values in terms of the coördinates $q_1, q_2, \cdots$ and the generalized momenta $p_1, p_2, \cdots$, where $p_1 = \frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}$, $p_2 = \frac{\partial T_{\dot{q}}}{\partial \dot{q}_2}$, etc.

Let us now try the experiment of replacing in $T_{\dot{q}}$ one only of the velocities $\dot{q}_1$ by its value in terms of the corresponding momentum p_1, the coördinates $q_1, q_2, \cdots$, and the remaining velocities $\dot{q}_2, \dot{q}_3, \cdots$.

Call $T_{\dot{q}}$ thus changed in form, T_{p_1}. Of course

$$T_{p_1} \equiv T_{\dot{q}}, \quad \text{and} \quad \dot{q}_1 = F(p_1, \dot{q}_2, \dot{q}_3, \cdots, q_1, q_2, \cdots).$$

We have $$\frac{\partial T_{p_1}}{\partial q_1} = \frac{\partial T_{\dot{q}}}{\partial q_1} + \frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_1}.$$

Transposing, $$\frac{\partial T_{\dot{q}}}{\partial q_1} = \frac{\partial T_{p_1}}{\partial q_1} - p_1\frac{\partial \dot{q}_1}{\partial q_1} = \frac{\partial}{\partial q_1}[T_{p_1} - p_1\dot{q}_1].$$

Again, $$\frac{\partial T_{p_1}}{\partial p_1} = \frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial p_1} = p_1\frac{\partial \dot{q}_1}{\partial p_1} = \frac{\partial}{\partial p_1}(p_1\dot{q}_1) - \dot{q}_1.$$

Hence $$-\dot{q}_1 = \frac{\partial}{\partial p_1}[T_{p_1} - p_1\dot{q}_1].$$

$$M_{q_1} = T_{p_1} - p_1\dot{q}_1$$

is called the *Lagrangian expression for the kinetic energy modified for the coördinate* q_1.

Our Lagrangian equation

$$\frac{d}{dt}\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}-\frac{\partial T_{\dot{q}}}{\partial q_1}=Q_1$$

becomes

$$\dot{p}_1-\frac{\partial M_{q_1}}{\partial q_1}=Q_1. \tag{1}$$

We have also

$$\dot{q}_1=-\frac{\partial M_{q_1}}{\partial p_1}. \tag{2}$$

It is noteworthy that (1) and (2) differ from the Hamiltonian equations for q_1 only in that the negative of the modified expression M_{q_1} appears in place of the Hamiltonian expression T_p.

Let us go on to the other coördinates.

$$\frac{\partial T_{p_1}}{\partial \dot{q}_2}=\frac{\partial T_{\dot{q}}}{\partial \dot{q}_2}+\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial \dot{q}_2},$$

whence

$$\frac{\partial T_{\dot{q}}}{\partial \dot{q}_2}=\frac{\partial T_{p_1}}{\partial \dot{q}_2}-p_1\frac{\partial \dot{q}_1}{\partial \dot{q}_2}=\frac{\partial}{\partial \dot{q}_2}[T_{p_1}-p_1\dot{q}_1]=\frac{\partial M_{q_1}}{\partial \dot{q}_2}.$$

$$\frac{\partial T_{p_1}}{\partial q_2}=\frac{\partial T_{\dot{q}}}{\partial q_2}+\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\frac{\partial \dot{q}_1}{\partial q_2},$$

whence

$$\frac{\partial T_{\dot{q}}}{\partial q_2}=\frac{\partial T_{p_1}}{\partial q_2}-p_1\frac{\partial \dot{q}_1}{\partial q_2}=\frac{\partial}{\partial q_2}[T_{p_1}-p_1\dot{q}_1]=\frac{\partial M_{q_1}}{\partial q_2}.$$

The Lagrangian equation for q_2 is therefore

$$\frac{d}{dt}\frac{\partial M_{q_1}}{\partial \dot{q}_2}-\frac{\partial M_{q_1}}{\partial q_2}=Q_2, \tag{3}$$

and differs from the ordinary form of the Lagrangian equation only in that $T_{\dot{q}}$ is replaced by the modified expression M_{q_1}.

In forming the modified expression it must be noted that $\dot{q}_1$ must be replaced by its value in terms of $p_1, \dot{q}_2, \dot{q}_3, \cdots, q_1, q_2, \cdots$, not only in $T_{\dot{q}}$ but in the term $p_1\dot{q}_1$ as well.

An advantage of the modified form is that when it has once been formed we can get by its aid Hamiltonian equations for one coördinate and Lagrangian equations for the others.

The reasoning just given can be extended easily to the case where we wish Hamiltonian equations for more than one coördinate and Lagrangian equations for the rest.

The results may be formulated as follows: Let $T_{p_1, p_2, \cdots, p_r}$ be the form assumed by $T_{\dot q}$ when $\dot q_1, \dot q_2, \cdots, \dot q_r$, are replaced by their values in terms of $p_1, p_2, \cdots, p_r, \dot q_{r+1}, \dot q_{r+2}, \cdots, \dot q_n, q_1, q_2, \cdots, q_n$. Then, if

$$M_{q_1, q_2, \cdots, q_r} = T_{p_1, p_2, \cdots, p_r} - p_1\dot q_1 - p_2\dot q_2 - \cdots - p_r\dot q_r,$$

we have equations of the type

$$\dot p_k - \frac{\partial M_{q_1, q_2, \cdots, q_r}}{\partial q_k} = Q_k,$$

$$\dot q_k = - \frac{\partial M_{q_1, q_2, \cdots, q_r}}{\partial p_k},$$

if $k < r+1$;

and
$$\frac{d}{dt}\frac{\partial M_{q_1, q_2, \cdots, q_r}}{\partial \dot q_k} - \frac{\partial M_{q_1, q_2, \cdots, q_r}}{\partial q_k} = Q_k,$$

if $k > r$.

18. If we modify the Lagrangian expression for the kinetic energy for all the coördinates,

$$M_{q_1, \cdots, q_n} = T_p - p_1\dot q_1 - p_2\dot q_2 - \cdots - p_n\dot q_n;$$

and we get Hamiltonian equations of the form

$$\dot p_k - \frac{\partial M_{q_1, \cdots, q_n}}{\partial q_k} = Q_k, \tag{1}$$

$$\dot q_k = - \frac{\partial M_{q_1, \cdots, q_n}}{\partial p_k}, \tag{2}$$

for all the coördinates, and as we have nowhere assumed in our reasoning that $T_{\dot q}$ is a homogeneous quadratic in the generalized velocities, we can use these equations safely when the geometrical equations contain the time explicitly (v. Art. 7).

If the geometrical equations do not involve the time, in which case $T_{\dot q}$ is a homogeneous quadratic in $\dot q_1, \dot q_2, \cdots,$

$$2\,T_{\dot q} = p_1\dot q_1 + p_2\dot q_2 + \cdots + p_n\dot q_n$$

by Euler's Theorem; and $M_{q_1,\cdots,q_n} = T_p - 2\,T_p = -\,T_p$; and (1) and (2) assume the familiar forms

$$\dot{p}_k + \frac{\partial T_p}{\partial q_k} = Q_k, \tag{3}$$

$$\dot{q}_k = \frac{\partial T_p}{\partial p_k}. \tag{4}$$

It is important to note that the modified Lagrangian expression $M_{q_1, q_2, \cdots, q_r}$ is not usually the *kinetic energy* of the system, although, as we shall see later, in some special problems it reduces to the kinetic energy. As we have just seen, when the time does not enter the geometrical equations, the completely modified Lagrangian expression (that is, the Lagrangian expression modified for all the coördinates) is the negative of the energy.

19. As an illustration of the employment of the Hamiltonian equations when the geometrical equations contain the time, let us take the tractrix problem of Art. 6, (a').

Here $$T_{\dot{q}} = \frac{m}{2}\left[n^2 + a^2\dot{\theta}^2 - 2\,an\cos\theta\dot{\theta}\right]$$

and is not homogeneous in $\dot{\theta}$.

$$p_\theta = \frac{\partial T_{\dot{q}}}{\partial \dot{\theta}} = m\left[a^2\dot{\theta} - an\cos\theta\right],$$

and $$\dot{\theta} = \frac{p_\theta}{a^2 m} + \frac{n}{a}\cos\theta. \tag{1}$$

$$T_p = \frac{1}{2}\left[mn^2\sin^2\theta + \frac{p_\theta^2}{a^2 m}\right].$$

$$M_\theta = T_p - p_\theta\dot{\theta} = \frac{mn^2\sin^2\theta}{2} - \frac{p_\theta^2}{2\,a^2 m} - \frac{n}{a}\cos\theta p_\theta.$$

$$\frac{\partial M_\theta}{\partial \theta} = mn^2\sin\theta\cos\theta + \frac{n}{a}\sin\theta p_\theta;$$

and we have $$\dot{p}_\theta - mn^2\sin\theta\cos\theta - \frac{n}{a}\sin\theta p_\theta = 0; \tag{2}$$

and (1) and (2) are our required Hamiltonian equations. Let us solve them.

From (1), $$p_\theta = ma^2\dot\theta - mna\cos\theta,$$

whence $$\dot p_\theta = ma^2\ddot\theta + mna\sin\theta\dot\theta.$$

Substituting in (2),

$$ma^2\ddot\theta + mna\sin\theta\dot\theta - mn^2\sin\theta\cos\theta - mna\sin\theta\dot\theta + mn^2\sin\theta\cos\theta = 0,$$

or $$\ddot\theta = 0,$$

which agrees with the result of Art. 6, (a').

EXAMPLES

1. Work Art. 6, (b'), by the Hamiltonian method.

2. Work Exs. 1 and 2, Art. 6, by the Hamiltonian method.

20. (a) As an example of the employment of the modified form, we shall take the tractrix problem of Art. 6 and modify for the coördinate x.

We have (v. Art. 6)

$$T_{\dot q} = \frac{m}{2}[\dot x^2 + a^2\dot\theta^2 - 2a\cos\theta\dot x\dot\theta].$$

$$p_x = m[\dot x - a\cos\theta\dot\theta],$$

$$\dot x = \frac{p_x}{m} + a\cos\theta\dot\theta,$$

$$T_{p_x} = \frac{m}{2}\left[\frac{p_x^2}{m^2} + a^2\sin^2\theta\dot\theta^2\right].$$

$$M_x = T_{p_x} - p_x\dot x = \frac{m}{2}\left[\frac{p_x^2}{m^2} + a^2\sin^2\theta\dot\theta^2\right] - \frac{p_x^2}{m^2} - a\cos\theta\dot\theta p_x$$

$$= \frac{m}{2}\left[a^2\sin^2\theta\dot\theta^2 - \frac{p_x^2}{m^2} - \frac{2}{m}a\cos\theta\dot\theta p_x\right].$$

$$\frac{\partial M_x}{\partial x} = 0,$$

$$\frac{\partial M_x}{\partial\dot\theta} = ma^2\sin^2\theta\dot\theta - a\cos\theta p_x,$$

$$\frac{\partial M_x}{\partial\theta} = ma^2\sin\theta\cos\theta\dot\theta^2 + a\sin\theta\dot\theta p_x.$$

We have for x the Hamiltonian equations

$$\dot{p}_x = R \sin\theta, \tag{1}$$

$$\dot{x} = \frac{p_x}{m} + a\cos\theta\dot{\theta}, \tag{2}$$

and for θ the Lagrangian equation

$$ma^2[\sin^2\theta\ddot{\theta} + \sin\theta\cos\theta\dot{\theta}^2] - a\cos\theta\dot{p}_x = 0. \tag{3}$$

Of course (1), (2), and (3) must be solved as simultaneous equations, and we can simplify by the aid of the condition

$$x = nt. \tag{4}$$

Solving, we get $\ddot{\theta} = 0,$

$$R = \frac{mn^2}{a}. \quad \text{(v. Art. 6 and Art. 15, } (c))$$

(*b*) As a second example we shall take the problem of the two particles and the table with a hole in it (v. Art. 8, (*d*)) and modify for θ.

We have $T_{\dot{q}} = \frac{m}{2}[2\dot{x}^2 + (a-x)\dot{\theta}^2].$

$$p_\theta = \frac{\partial T_{\dot{q}}}{\partial\dot{\theta}} = m(a-x)^2\dot{\theta},$$

whence

$$\dot{\theta} = \frac{p_\theta}{m(a-x)^2}. \tag{1}$$

$$T_{p_\theta} = \frac{m}{2}\left[2\dot{x}^2 + \frac{p_\theta^2}{m^2(a-x)^2}\right].$$

$$M_\theta = \frac{m}{2}\left[2\dot{x}^2 + \frac{p_\theta^2}{m^2(a-x)^2}\right] - \frac{p_\theta^2}{m(a-x)^2}$$

$$= \frac{m}{2}\left[2\dot{x}^2 - \frac{p_\theta^2}{m^2(a-x)^2}\right].$$

Our Hamiltonian equations are (1) and

$$\dot{p}_\theta = 0. \tag{2}$$

Our Lagrangian equation is

$$m\left[2\ddot{x}+\frac{p_\theta^2}{m^2(a-x)^3}\right]=mg. \qquad (3)$$

By (2), $$p_\theta = C = ma\sqrt{ag};$$

whence (3) becomes $$2\ddot{x}+\frac{a^3g}{(a-x)^3}=g, \qquad (4)$$
as in Art. 8, (*d*).

(*c*) As a third example we shall take the wedge and sphere problem of Ex. 3, Art. 9, and modify for x.

We have

$$T_{\dot{q}}=\frac{1}{2}(M+m)\dot{x}^2+\frac{m}{2}\left[\frac{a^2+k^2}{a^2}\dot{y}^2-2\dot{x}\dot{y}\cos\alpha\right].$$

$$p_x=\frac{\partial T_{\dot{q}}}{\partial\dot{x}}=(M+m)\dot{x}-m\dot{y}\cos\alpha.$$

$$\dot{x}=\frac{p_x+m\dot{y}\cos\alpha}{M+m}. \qquad (1)$$

$$T_{p_x}=\frac{p_x^2-m^2\dot{y}^2\cos^2\alpha}{2(M+m)}+\frac{m}{2}\frac{a^2+k^2}{a^2}\dot{y}^2.$$

$$M_x=\frac{p_x^2-m^2\dot{y}^2\cos^2\alpha}{2(M+m)}+\frac{m}{2}\frac{a^2+k^2}{a^2}\dot{y}^2-\frac{p_x^2+mp_x\dot{y}\cos\alpha}{M+m}.$$

Our Hamiltonian equations are (1) and

$$\dot{p}_x=0. \qquad (2)$$

Our Lagrangian equation is

$$\left[m\frac{a^2+k^2}{a^2}-\frac{m^2\cos^2\alpha}{M+m}\right]\ddot{y}-\frac{mp_x\cos\alpha}{M+m}=mg\sin\alpha. \qquad (3)$$

By (2), $$p_x=C=0; \qquad (4)$$

whence (3) becomes $$\left[\frac{a^2+k^2}{a^2}-\frac{m\cos^2\alpha}{M+m}\right]\ddot{y}=g\sin\alpha, \qquad (5)$$

as in Art. 9, Ex. 3.

(*d*) As a fourth example we shall take the flexible-parallelogram problem (v. Art. 9, (*b*)) and modify for ϕ.

We have
$$T_{\dot q} = \frac{m}{2}[2k^2\dot\phi^2 + a^2\dot\theta^2].$$

$$p_\phi = \frac{\partial T_{\dot q}}{\partial\dot\phi} = 2mk^2\dot\phi.$$

$$\dot\phi = \frac{p_\phi}{2mk^2}. \tag{1}$$

$$T_{p_\phi} = \frac{m}{2}\left[\frac{p_\phi^2}{2m^2k^2} + a^2\dot\theta^2\right].$$

$$M_\phi = \frac{m}{2}\left[a^2\dot\theta^2 - \frac{p_\phi^2}{2m^2k^2}\right].$$

Our Hamiltonian equations are (1) and

$$\dot p_\phi = 0. \tag{2}$$

Our Lagrangian equation is

$$ma^2\ddot\theta = -mga\sin\theta, \tag{3}$$

or
$$\ddot\theta + \frac{g}{a}\sin\theta = 0. \tag{4}$$

EXAMPLES

1. Take the dumb-bell problem of Art. 8, (*c*), and modify for θ.

Ans.
$$T_{p_\theta} = m\left[\dot x^2 + \dot y^2 + \frac{p_\theta^2}{m^2a^2}\right].$$

$$M_\theta = m\left[\dot x^2 + \dot y^2 - \frac{p_\theta^2}{m^2a^2}\right].$$

$$\dot p_\theta = 0.$$

$$m\ddot x = 0.$$

$$m\ddot y = 0.$$

2. Take the dumb-bell problem of Art. 8, (*c*), and modify for x and y.

Ans. $$T_{p_x, p_y} = \frac{m}{4}\left[\frac{p_x^2 + p_y^2}{m^2} + a^2\dot{\theta}^2\right].$$

$$M_{x,y} = \frac{m}{4}\left[a^2\dot{\theta}^2 - \frac{p_x^2 + p_y^2}{m^2}\right].$$

$$\dot{p}_x = 0.$$

$$\dot{p}_y = 0.$$

$$\tfrac{1}{2}\, ma^2\ddot{\theta} = 0.$$

3. Take the gyroscope problem, Art. 10, (*c*), and modify for ϕ and ψ.

$$T_{p_\phi, p_\psi} = \frac{1}{2}\left[A\dot{\theta}^2 + \frac{p_\phi^2}{C} + \frac{(p_\psi - p_\phi \cos\theta)^2}{A\sin^2\theta}\right].$$

$$M_{\phi,\psi} = \frac{1}{2}\left[A\dot{\theta}^2 + \frac{p_\phi^2}{C} - \frac{(p_\psi - p_\phi \cos\theta)^2}{A\sin^2\theta}\right].$$

$$\dot{p}_\phi = 0, \qquad \dot{\phi} = \frac{p_\phi}{C} - \frac{(p_\psi - p_\phi\cos\theta)\cos\theta}{A\sin^2\theta}.$$

$$\dot{p}_\psi = 0, \qquad \dot{\psi} = \frac{p_\psi - p_\phi\cos\theta}{A\sin^2\theta}.$$

$$A\ddot{\theta} - \frac{(p_\phi^2 + p_\psi^2)\cos\theta - p_\phi p_\psi(1 + \cos^2\theta)}{A\sin^2\theta} = mga\sin\theta.$$

21. We proceed to comment on the problems of the preceding section.

(*a*) No one of them involves the time explicitly in the geometrical equations, and therefore the kinetic energy $T_{\dot{q}}$ in all of them is a homogeneous quadratic in the generalized velocities.

(*b*) The momenta, therefore, are homogeneous of the first degree in the velocities, and consequently the eliminated velocities are homogeneous of the first degree in the corresponding momenta and the remaining velocities, and the energy $T_{p_1, p_2, \ldots}$ and the modified function $M_{q_1, q_2, \ldots}$ are homogeneous quadratics in the introduced momenta and the velocities not eliminated.

(c) In all the problems the coördinates for which we have modified the Lagrangian expression for the kinetic energy are *cyclic.* In all of them except the first no work is done when any one of the coördinates in question is varied. Consequently one of the Hamiltonian equations for that coördinate is of the form $\dot{p}_k = 0$, and the momentum $p_k = c_k$, where c_k is a constant. Therefore it is easy to express the energy $T_{p_1, p_2}, \ldots$ and the modified expression $M_{q_1, q_2}, \ldots$ in terms of the remaining coördinates, the corresponding velocities, and the constants $c_1, c_2, \cdots$, and when so expressed they are quadratics in the velocities but not necessarily homogeneous quadratics. When the modified function has been so expressed, it may be used in forming the Lagrangian equations for the remaining coördinates precisely as the Lagrangian expression for the kinetic energy is used, and the coördinates that have been eliminated may be *ignored* in the rest of the work of solving the problem unless we are interested in their values (v. Art. 16).

(d) To generalize: If some of the coördinates of a moving system are *cyclic*, and if the impressed forces are such that when any one of these coördinates is varied no work is done, the momenta corresponding to these coördinates are constant throughout the motion. The substitution of these constants for the momenta in the Lagrangian expression for the kinetic energy modified for the coördinates in question will reduce it to an explicit function of the remaining coördinates, the corresponding velocities, and the constants substituted, which will be a quadratic in the velocities but not necessarily a homogeneous quadratic.

When the modified function has been so expressed, it may be used in forming the Lagrangian equations for the remaining coördinates precisely as the Lagrangian expression for the kinetic energy is used, and the coördinates that have been eliminated may be *ignored* in the rest of the work of solving the problem (v. Art. 16).

In the important case where the system starts from rest, the constant momenta corresponding to the ignorable coördinates

being zero at the start are zero throughout the motion, and the modified expression is identical with the Lagrangian expression for the kinetic energy, which therefore is a function of the remaining coördinates and the corresponding velocities and is a homogeneous quadratic in the velocities (v. Art. 24, (*a*)).

(*e*) The fact that the Lagrangian expression for the kinetic energy modified for ignorable coördinates is expressible in terms of the remaining coördinates and the corresponding velocities and is a quadratic in terms of those velocities is often of great importance, as we shall see later.

22. Let us take the gyroscope problem of Art. 10, (*c*), and Art. 20, Ex. 3, and work it from the start, ignoring the *cyclic* coördinates ϕ and ψ.

We have $T_{\dot q} = \frac{1}{2}[A(\dot\theta^2 + \sin^2\theta\dot\psi^2) + C(\dot\psi\cos\theta + \dot\phi)^2]$, and therefore ϕ and ψ are cyclic coördinates. Moreover, no work is done when ϕ is varied nor when ψ is varied, so that ϕ and ψ are ignorable.

$$\dot p_\phi = 0, \quad \text{and} \quad \dot p_\psi = 0, \quad \text{so that} \quad p_\phi = c_1, \quad \text{and} \quad p_\psi = c_2.$$

We have
$$p_\phi = \frac{\partial T_{\dot q}}{\partial\dot\phi} = C(\dot\psi\cos\theta + \dot\phi) = c_1,$$

$$p_\psi = \frac{\partial T_{\dot q}}{\partial\dot\psi} = A\sin^2\theta\dot\psi + C\cos\theta(\dot\psi\cos\theta + \dot\phi) = c_2,$$

whence
$$\dot\phi = \frac{c_1}{C} - \frac{(c_2 - c_1\cos\theta)\cos\theta}{A\sin^2\theta},$$

and
$$\dot\psi = \frac{c_2 - c_1\cos\theta}{A\sin^2\theta}.$$

$$T_{p_\phi p_\psi} = \frac{1}{2}\left[A\dot\theta^2 + \frac{c_1^2}{C} + \frac{(c_2 - c_1\cos\theta)^2}{A\sin^2\theta}\right].$$

$$M_{\phi\psi} = \frac{1}{2}\left[A\dot\theta^2 - \frac{c_1^2}{C} - \frac{(c_2 - c_1\cos\theta)^2}{A\sin^2\theta}\right].$$

Forming the Lagrangian equation for θ in the usual way, we have

$$A\ddot{\theta} - \frac{(c_1^2 + c_2^2)\cos\theta - c_1 c_2 (1 + \cos^2\theta)}{A \sin^3\theta} = mga \sin\theta,$$

or

$$A\ddot{\theta} + \frac{(c_1 - c_2\cos\theta)(c_2 - c_1\cos\theta)}{A \sin^3\theta} = mga \sin\theta,$$

which is identical with (7), Art. 10, (c).

EXAMPLE

Work the problem of Art. 20, (b), ignoring the coördinate θ.

23. The problem of Art. 20, (d), has an interesting peculiarity. Both coördinates are cyclic, and there is no term in the kinetic energy that is linear in $\dot{\phi}$, the velocity corresponding to the ignorable coördinate ϕ. Consequently the constant momentum p_ϕ is a constant multiple of $\dot{\phi}$, which is therefore itself a constant. This is true also of $p_\phi\dot{\phi}$ and of $mk^2\dot{\phi}^2$, the term in the energy which involves $\dot{\phi}$. T_{p_ϕ} and M_ϕ must then differ from $ma^2\dot{\theta}^2$ by constants, and as only the derivatives of M_ϕ are used in forming the Lagrangian equation for θ, we have merely to disregard the term $mk^2\dot{\phi}^2$ in the energy $T_{\dot{q}}$ and use what is left of $T_{\dot{q}}$ instead of M_ϕ.

In cases like this we are able practically to ignore the contribution to the energy made by the ignored coördinate as well as the coördinate itself, and of course we can conclude that the motion we may thus disregard has no effect on the motion we are studying, but that the two can go on together without interference.

EXAMPLE

Examine Exs. 1 and 2, Art. 20, from the point of view of the present article.

24. (a) The wedge and sphere problem of Art. 20, (c), belongs to a very important class. Both x and y are cyclic coördinates, and x is ignorable.

$$p_x = \frac{\partial T_{\dot{q}}}{\partial \dot{x}} = (M + m)\dot{x} - m\dot{y}\cos\alpha, \qquad \dot{p}_x = 0.$$

The momentum p_x is constant, and as it is initially zero (since the system starts from rest) it is zero throughout the motion, as is $p_x\dot{x}$. Consequently the kinetic energy T_{p_x} and the modified expression M_x are identical, and as they are both homogeneous quadratics in $\dot{y}$ and p_x and do not contain y, they reduce to the form $L\dot{y}^2$, where L is a constant.

Therefore our Lagrangian equation for y is $L\ddot{y} = mg \sin \alpha$, and the sphere rolls down the wedge with constant acceleration. If we care only for the motion of the sphere on the wedge, we may then ignore x completely and yet know enough of the form of M_x to get valuable information as to the required motion. Of course we know that the energy of the whole system can be expressed in terms of $\dot{y}$, and if we are able by any means to so express it, we can solve completely for y without using the ignored coördinate x at any stage of the process.

(*b*) As a striking example of this complete *ignoration* of coördinates, and of dealing with a moving system having an infinite number of degrees of freedom, let us take the motion of a homogeneous sphere under gravity in an infinite incompressible liquid, both sphere and liquid being initially at rest.

From considerations of symmetry, the position of the sphere can be fixed by giving a single coördinate x, the distance of the center of the sphere below a fixed level, and x is clearly a cyclic coördinate.

The positions of the particles of the liquid can be given in terms of x and a sufficiently large number (practically infinite) of coördinates $q_1, q_2, \cdots$, in a great variety of ways. Assume that a set has been chosen such that all the q's are cyclic.* Then, since gravity does no work unless the position of the sphere is varied, the q's are all ignorable. That is, for every one of them $\dot{p}_k = 0$, and the momentum $p_k = c_k$, and since the system starts from rest the initial value of p_k is zero, and therefore $c_k = 0$. $M_{q_1, q_2, \ldots}$, the energy of the system modified for all the q's,

* That this is possible will be shown later, in connection with the treatment of Impulsive Forces (v. Chap. III, Art. 36).

is then identical with the energy $T_{p_1, p_2, \ldots}$ and must be expressible in terms of the remaining coördinate x and the corresponding velocity $\dot{x}$ (v. Art. 21, (*d*)); and as x is cyclic the energy of the system will then be of the form $L\dot{x}^2$, where L is a constant.

Forming the Lagrangian equation for x, we have

$$L\ddot{x} = mg,$$

and we learn that the sphere will descend with constant acceleration.

Of course this brief solution is incomplete, as it gives no information as to the motion of the particles of the liquid, and since we do not know the value of L, we do not learn the magnitude of the acceleration. Still the solution is interesting and valuable.

The energy of the moving liquid, calculated by the aid of hydromechanics, proves to be $\frac{1}{2} m'\dot{x}^2$, where m' is one half the mass of the liquid displaced by the sphere (Lamb, Hydromechanics, Art. 91, (3)), and therefore the energy of the system is $\frac{1}{2}(m + m')\dot{x}^2$, and this agrees with our result.

25. Summary of Chapter II. The kinetic energy of a moving system which has n degrees of freedom can be expressed in terms of the n coördinates $q_1, q_2, \cdots, q_n$, and the n generalized momenta $p_1, p_2, \cdots, p_n$, and when so expressed it is a homogeneous quadratic in the momenta if the geometrical equations do not involve the time (v. Art. 14), and is called the Hamiltonian expression for the kinetic energy. If T_p is the Hamiltonian expression for the kinetic energy, and $T_{\dot{q}}$ the Lagrangian expression for the kinetic energy,

$$\frac{\partial T_p}{\partial q_k} = -\frac{\partial T_{\dot{q}}}{\partial q_k}, \quad \text{and} \quad \frac{\partial T_p}{\partial p_k} = \dot{q}_k.$$

The work done by the effective forces in a hypothetical infinitesimal displacement of the system, due to an infinitesimal change δq_k in a single coördinate q_k, is $\left[\dot{p}_k + \frac{\partial T_p}{\partial q_k}\right]\delta q_k$. If this be written equal to $Q_k \delta q_k$, the work done by the actual

forces in the displacement in question, there will result the differential equation of the first order,

$$\dot{p}_k + \frac{\partial T_p}{\partial q_k} = Q_k.$$

The $2n$ equations of which this and

$$\frac{\partial T_p}{\partial p_k} = \dot{q}_k,$$

given above, are the type are known as the Hamiltonian equations of motion for the system.

If, in the expression $T_{\dot{q}} - p_1\dot{q}_1 - p_2\dot{q}_2 - \cdots - p_r\dot{q}_r$, $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_r$, are replaced by their values in terms of $p_1, p_2, \cdots, p_r, \dot{q}_{r+1}, \cdots, \dot{q}_n, q_1, q_2, \cdots, q_n$, the result $M_{q_1,\ldots,q_r}$ is the Lagrangian expression for the kinetic energy modified for the coördinates $q_1, \cdots, q_r$. For the coördinates for which the expression has been modified, that is, for $k < r+1$, we have Hamiltonian equations of the type

$$\dot{p}_k - \frac{\partial M}{\partial q_k} = Q_k,$$
$$\dot{q}_k = -\frac{\partial M}{\partial p_k}.$$

For the remaining coördinates, that is, for $k > r$, we have Lagrangian equations of the type

$$\frac{d}{dt}\frac{\partial M}{\partial \dot{q}_k} - \frac{\partial M}{\partial q_k} = Q_k. \qquad \text{(v. Art. 17)}$$

Whether we work from the Lagrangian expression $T_{\dot{q}}$ for the kinetic energy, or the Hamiltonian expression T_p, or the modified Lagrangian expression $M_{q_1,\ldots}$, we are in general led to a set of simultaneous differential equations whose number depends merely upon the number of degrees of freedom in the moving system, and such that to solve one we must form and solve all.

If, however, some of the coördinates are *cyclic* (v. Art. 16), and the impressed forces are such that when any one of them

is varied no work is done, the corresponding momenta are constant throughout the motion, and these coördinates are ignorable in the sense that if constants are substituted for the corresponding momenta in T_p or $M_{q_1}, \ldots$, the Hamiltonian equations for the remaining coördinates in the former case (or the Lagrangian equations in the latter case) can be formed and, if capable of solution, can be solved without forming the equations corresponding to the ignored coördinates.

If the system starts from rest and there are ignorable coördinates, $M_{q_1}, \ldots$, the Lagrangian expression modified for the ignorable coördinates, is identical with the kinetic energy of the system; and whether the system starts from rest or not, $M_{q_1}, \ldots$ is a quadratic, but not necessarily a homogeneous quadratic, in the velocities corresponding to the coördinates which are not ignorable.

CHAPTER III

IMPULSIVE FORCES

26. Virtual Moments. If a hypothetical infinitesimal displacement is given to a system, the product of any force by the distance its point of application is moved in the direction of the force is called the *virtual moment* of the force, and the sum of all the virtual moments is called the virtual moment of the set of forces.

If the forces are finite forces, the virtual moment is the *virtual work*; that is, the work which would be done by the forces in the assumed displacement. If the forces are impulsive forces, the virtual moment is not virtual work but has an interpretation as *virtual action*, which we shall give later when we take up what is called the *action* of a moving system.

27. For the motion of a particle under impulsive forces we have the familiar equations

$$m(\dot{x}_1 - \dot{x}_0) = X,$$

$$m(\dot{y}_1 - \dot{y}_0) = Y,$$

$$m(\dot{z}_1 - \dot{z}_0) = Z.$$

$$m(\dot{x}_1 - \dot{x}_0), \qquad m(\dot{y}_1 - \dot{y}_0), \qquad m(\dot{z}_1 - \dot{z}_0),$$

are called the *effective* impulsive forces on the particle and are mechanically equivalent to the actual forces.

If the point is given an infinitesimal displacement,

$$m[(\dot{x}_1 - \dot{x}_0)\,\delta x + (\dot{y}_1 - \dot{y}_0)\,\delta y + (\dot{z}_1 - \dot{z}_0)\,\delta z]$$

is the virtual moment of the effective forces and of course is equal to the virtual moment of the actual forces.

If the generalized coördinates of a moving system acted on by impulsive forces are q_1, q_2, $\cdots$, and a displacement caused by varying q_1 by δq_1 is given to the system,

$$\begin{aligned}\delta_{q_1}A &= \Sigma m\left[(\dot{x}_1-\dot{x}_0)\,\delta_{q_1}x+(\dot{y}_1-\dot{y}_0)\,\delta_{q_1}y+(\dot{z}_1-\dot{z}_0)\,\delta_{q_1}z\right]\\ &= \Sigma m\left[(\dot{x}_1-\dot{x}_0)\frac{\partial x}{\partial q_1}+(\dot{y}_1-\dot{y}_0)\frac{\partial y}{\partial q_1}+(\dot{z}_1-\dot{z}_0)\frac{\partial z}{\partial q_1}\right]\delta q_1,\end{aligned}$$

where $\delta_{q_1}A$ represents the virtual moment of the effective forces.

As in Art. 7, $$\frac{\partial \dot{x}}{\partial \dot{q}_1}=\frac{\partial x}{\partial q_1}.$$

Therefore $$m\dot{x}\frac{\partial x}{\partial q_1}=m\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1}=\frac{m}{2}\frac{\partial}{\partial \dot{q}_1}(\dot{x}^2),$$

and $$\delta_{q_1}A=\left[\left(\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\right)_1-\left(\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\right)_0\right]\delta q_1.$$

Hence $$\left[\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\right]_1-\left[\frac{\partial T_{\dot{q}}}{\partial \dot{q}_1}\right]_0=P_{q_1} \qquad (1)$$

(where $P_{q_1}\delta q_1$ is the virtual moment of the impressed impulsive forces and P_{q_1} is called the *component of impulse* corresponding to q_1) is our Lagrangian equation, and of course we have one such equation for every coördinate q_k.

Equation (1) can be written in the equivalent form

$$(p_1)_1-(p_1)_0=P_{q_1}. \qquad (2)$$

28. Illustrative Examples. (*a*) A lamina of mass m rests on a smooth horizontal table and is acted on by an impulsive force of magnitude P in the plane of the lamina. Find the initial motion.

Let (x, y) be the center of gravity of the lamina, let θ be the angle made with the axis of X by a perpendicular to the line of action of the force, and let a be the distance of the force from the center of gravity.

Then $$T_{\dot q} = \frac{m}{2}[\dot x^2 + \dot y^2 + k^2\dot\theta^2]. \quad \text{(v. App. A, § 10)}$$

$$p_x = \frac{\partial T_{\dot q}}{\partial \dot x} = m\dot x,$$

$$p_y = \frac{\partial T_{\dot q}}{\partial \dot y} = m\dot y,$$

$$p_\theta = \frac{\partial T_{\dot q}}{\partial \dot\theta} = mk^2\dot\theta.$$

$$(p_x)_0 = (p_y)_0 = (p_\theta)_0 = 0.$$

If the axes are chosen so that $x_0 = 0$, $y_0 = 0$, and $\theta_0 = 0$, we have

$$m\dot x = 0,$$

$$m\dot y = P,$$

$$mk^2\dot\theta = aP.$$

Hence the initial velocities are

$$\dot x = 0, \qquad \dot y = \frac{P}{m}, \qquad \dot\theta = \frac{aP}{mk^2}.$$

The velocities of a point on the axis of X at the distance b from the center of gravity are

$$v_x = 0,$$
$$v_y = \dot y + b\dot\theta = \left[1 + \frac{ab}{k^2}\right]\frac{P}{m}.$$

The point in question will have no initial velocity if $b = -\frac{k^2}{a}$.

It follows that the lamina begins to rotate about an instantaneous center in the perpendicular from the center of gravity to the line of action of the force at a distance $\frac{a^2 + k^2}{a}$ from that line and situated on the same side of the line of the force as the center of gravity. This point is called the *center of percussion.*

(*b*) A wedge of angle α and mass M, smooth below and perfectly rough above, rests on a horizontal plane.

A sphere of radius a and mass m is rotating with angular velocity Ω about a horizontal axis parallel to the edge of the wedge and is placed gently on the wedge. Find the initial motion (v. Art. 9, Ex. 3).

Take as coördinates x, the distance of the edge of the wedge from a fixed axis parallel to it in the horizontal plane; y, the distance of the point of contact of the sphere down the wedge; and θ, the angle through which the sphere has rotated.

We have

$$T_{\dot{q}} = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}(\dot{y}^2 - 2\,\dot{x}\dot{y}\cos\alpha + k^2\dot{\theta}^2),$$

both before and after the sphere is set down. After the sphere is set down, $\dot{y} - a\dot{\theta} = 0$. Before it is set down, $\dot{x} = \dot{y} = 0$ and $\dot{\theta} = \Omega$. Since the sphere cannot slip, it exerts an impulsive force P up the wedge, and an equal and opposite force P is exerted on it by the wedge at the instant the two bodies come in contact.

We have $\quad p_x = \dfrac{\partial T_{\dot{q}}}{\partial \dot{x}} = (M+m)\,\dot{x} - m\dot{y}\cos\alpha,$

$$p_y = \frac{\partial T_{\dot{q}}}{\partial \dot{y}} = m\,(\dot{y} - \dot{x}\cos\alpha),$$

$$p_\theta = \frac{\partial T_{\dot{q}}}{\partial \dot{\theta}} = mk^2\dot{\theta}.$$

Our equations are

$$(M+m)\,\dot{x} - m\dot{y}\cos\alpha = 0, \tag{1}$$

$$m\,(\dot{y} - \dot{x}\cos\alpha) = P, \tag{2}$$

$$mk^2(\dot{\theta} - \Omega) = -\,aP, \tag{3}$$

and we have also $\quad a\dot{\theta} = \dot{y}. \quad (4)$

From (2), (3), and (4),

$$\frac{a^2+k^2}{a^2}\,\dot{y} - \dot{x}\cos\alpha = \frac{k^2\Omega}{a}. \tag{5}$$

From (1) and (5),

$$\left[(M+m)\frac{a^2+k^2}{a^2}-m\cos^2\alpha\right]\dot{x}=\frac{mk^2}{a}\,\Omega\cos\alpha,$$

and $$\left[(M+m)\frac{a^2+k^2}{a^2}-m\cos^2\alpha\right]\dot{y}=(M+m)\frac{k^2}{a}\,\Omega.$$

(*c*) In Art. 8, (*b*), let the weight $4\,m$ be jerked down with a velocity v. Find the initial motion. Take the coördinates x and y as in Art. 8, (*b*), and let P be the magnitude of the jerk.

$$T_{\dot{q}}=\frac{m}{2}(8\,\dot{x}^2+3\,\dot{y}^2-2\,\dot{x}\dot{y}).$$

$$p_x=\frac{\partial T_{\dot{q}}}{\partial\dot{x}}=m\,(8\,\dot{x}-\dot{y}),$$

$$p_y=\frac{\partial T_{\dot{q}}}{\partial\dot{y}}=m\,(3\,\dot{y}-\dot{x}).$$

Our equations are $m\,(8\,\dot{x}-\dot{y})=P,$

and $m\,(3\,\dot{y}-\dot{x})=0.$

Whence $23\,m\dot{x}=3\,P,$

$$23\,m\dot{y}=P.$$

But $$\dot{x}=v=\frac{3}{23}\frac{P}{m},$$

and $$\dot{y}=\tfrac{1}{3}\,v.$$

(*d*) Four equal rods freely jointed together in the form of a square are at rest on a horizontal table. A blow is struck at one corner in the direction of one of the sides. Compare the initial velocities of the middle points of the four rods.

Let m be the mass and $2\,a$ the length of a rod. Take a pair of rectangular axes in the table. Let $(x,\,y)$ be the center of the figure at any time and θ and ϕ the angles made by two adjacent rods with the axis of X. $x,\,y,\,\theta,\,\phi$, are our generalized coördinates.

The rectangular coördinates of the four middle points are obviously

$$(x-a\cos\phi,\ \ y-a\sin\phi), \tag{1}$$

$$(x+a\cos\theta,\ \ y+a\sin\theta), \tag{2}$$

$$(x + a\cos\phi,\ y + a\sin\phi), \tag{3}$$

$$(x - a\cos\theta,\ y - a\sin\theta). \tag{4}$$

We have

$$T_{\dot{q}} = \frac{m}{2}[4\dot{x}^2 + 4\dot{y}^2 + 2a^2(\dot{\theta}^2 + \dot{\phi}^2) + 2k^2(\dot{\theta}^2 + \dot{\phi}^2)].$$

$$p_x = 4m\dot{x},$$
$$p_y = 4m\dot{y},$$
$$p_\theta = 2m(a^2 + k^2)\dot{\theta},$$
$$p_\phi = 2m(a^2 + k^2)\dot{\phi}.$$

Let the values of x, y, θ, ϕ, be 0, 0, 0, $\frac{\pi}{2}$, before the blow is struck, and let P be the magnitude of the blow.

Our equations are
$$4m\dot{x} = P,$$
$$4m\dot{y} = 0,$$
$$2m(a^2 + k^2)\dot{\theta} = 0,$$
and
$$2m(a^2 + k^2)\dot{\phi} = aP.$$

Whence
$$\dot{y} = 0,$$
$$\dot{\theta} = 0,$$
$$\dot{x} = \frac{P}{4m},$$
$$\dot{\phi} = \frac{aP}{2m(a^2 + k^2)}.$$

Let v_1, v_2, v_3, v_4, be the required velocities of the four middle points. Then

$$v_1 = \dot{x} + a\dot{\phi} = \frac{3a^2 + k^2}{a^2 + k^2}\cdot\frac{P}{4m} = \frac{5}{8}\frac{P}{m},$$

$$v_2 = \dot{x} = \frac{P}{4m} = \frac{2}{8}\frac{P}{m},$$

$$v_3 = \dot{x} - a\dot{\phi} = \frac{k^2 - a^2}{a^2 + k^2}\frac{P}{4m} = -\frac{1}{8}\frac{P}{m},$$

$$v_4 = \dot{x} = \frac{P}{4m} = \frac{2}{8}\frac{P}{m},$$

and $v_1 : v_2 : v_3 : v_4 = 5 : 2 : -1 : 2.$

29. General Theorems. Work done by an Impulse. If a particle (x, y, z) initially at rest is displaced to the position $(x+\delta x, y+\delta y, z+\delta z)$, the displacement can be conceived of as brought about in the interval of time δt by imposing upon the particle a velocity whose components parallel to the axes are u_1, v_1, w_1, where $\delta x = u_1\delta t$, $\delta y = v_1\delta t$, and $\delta z = w_1\delta t$.

If the particle is initially in motion with a velocity whose components are u, v, w, the displacement in question could be brought about by imposing upon it an additional velocity whose components are obviously $u_1 - u$, $v_1 - v$, $w_1 - w$.

Let a moving system be acted on by a set of impulsive forces. Let m be the mass of any particle of the system; P_x, P_y, P_z, the components of the impulsive force acting on the particle; u, v, w, the components of the velocity of the particle before, and u_1, v_1, w_1 after, the impulsive forces have acted.

If any infinitesimal displacement is given to the system by which the coördinates of the particle are changed by δx, δy, and δz, we have the virtual moment of the effective forces equal to the virtual moment of the actual forces (v. Art. 27); that is,

$$\Sigma m\left[(u_1-u)\,\delta x+(v_1-v)\,\delta y+(w_1-w)\,\delta z\right] \\ =\Sigma\left[P_x\delta x+P_y\delta y+P_z\delta z\right].$$

If the velocity that would have to be imposed upon the particle m, were it at rest, to bring about its assumed displacement in the time δt has the components u_2, v_2, w_2, $\delta x = u_2\delta t$, $\delta y = v_2\delta t$, $\delta z = w_2\delta t$, and the equation above may be written

$$\Sigma m\left[(u_1-u)\,u_2+(v_1-v)\,v_2+(w_1-w)\,w_2\right] \\ =\Sigma\left[u_2P_x+v_2P_y+w_2P_z\right]. \qquad (1)$$

Interesting special cases of (1) are

$$\Sigma m\left[(u_1-u)\,u+(v_1-v)\,v+(w_1-w)\,w\right] \\ =\Sigma\left[uP_x+vP_y+wP_z\right], \qquad (2)$$

$$\Sigma m\left[(u_1-u)\,u_1+(v_1-v)\,v_1+(w_1-w)\,w_1\right] \\ =\Sigma\left[u_1P_x+v_1P_y+w_1P_z\right], \qquad (3)$$

the displacement used in (2) being what the system would have had in the time δt had the initial motion continued, and that in (3) what it has in the actual motion brought about by the impulsive forces. If we take half the sum of (2) and (3), we get

$$\sum \frac{m}{2}\left[u_1^2+v_1^2+w_1^2\right]-\sum \frac{m}{2}\left[u^2+v^2+w^2\right]$$
$$=\sum\left[\frac{u+u_1}{2}P_x+\frac{v+v_1}{2}P_y+\frac{w+w_1}{2}P_z\right]; \qquad (4)$$

or, *a system's gain in kinetic energy caused by the action of impulsive forces is the sum of the terms obtained by multiplying every force by half the sum of the initial and final velocities of its point of application, both being resolved in the direction of the force.*

This sum is usually called the *work* done by the impulsive forces.

30. Thomson's Theorem. If our system starts from rest, $u=v=w=0$, and formulas (1) and (3), Art. 29, reduce respectively to

$$\Sigma m\left[u_1u_2+v_1v_2+w_1w_2\right]=\Sigma\left[u_2P_x+v_2P_y+w_2P_z\right], \qquad (1)$$

and
$$\Sigma m\left[u_1^2+v_1^2+w_1^2\right]=\Sigma\left[u_1P_x+v_1P_y+w_1P_z\right]. \qquad (2)$$

But the first member of (1) is identically

$$\sum \frac{m}{2}\{[u_2^2+v_2^2+w_2^2]+[u_1^2+v_1^2+w_1^2]$$
$$-[(u_2-u_1)^2+(v_2-v_1)^2+(w_2-w_1)^2]\},$$

and subtracting (2) from (1) we get

$$\sum \frac{m}{2}\{[u_2^2+v_2^2+w_2^2]-[u_1^2+v_1^2+w_1^2]$$
$$-[(u_2-u_1)^2+(v_2-v_1)^2+(w_2-w_1)^2]\}$$
$$=\Sigma\left[(u_2-u_1)P_x+(v_2-v_1)P_y+(w_2-w_1)P_z\right]. \qquad (3)$$

If u_2, v_2, w_2, are the components of velocity of the particle m in any conceivable motion of the system which could give the

points of application of the impulsive forces the same velocities that they have in the actual motion, then the second member of (3) is zero and we get *Thomson's Theorem:*

If a system at rest is set in motion by impulsive forces, its kinetic energy is less than in any other motion where the velocities of the points of application of the forces in question are the same as in the actual motion, by an amount equal to the energy the system would have in the motion which, compounded with the actual motion, would produce the hypothetical motion.

31. Bertrand's Theorem. If Q_x, Q_y, and Q_z are the components of the impulsive force which would have to act on the particle m of the system considered in Art. 29 to change its component velocities from u, v, w, to u_2, v_2, w_2, formula (3), Art. 29, gives us

$$\Sigma m[(u_2 - u)u_2 + (v_2 - v)v_2 + (w_2 - w)w_2] = \Sigma[u_2Q_x + v_2Q_y + w_2Q_z]. \quad (1)$$

Subtracting (1) from (1) in Art. 29, we get

$$\Sigma m[(u_1u_2 + v_1v_2 + w_1w_2) - (u_2^2 + v_2^2 + w_2^2)] = \Sigma[u_2(P_x - Q_x) + v_2(P_y - Q_y) + w_2(P_z - Q_z)]. \quad (2)$$

The first member of (2) is (v. Art. 30) identically

$$\sum \frac{m}{2}\{[u_1^2 + v_1^2 + w_1^2] - [u_2^2 + v_2^2 + w_2^2] - [(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2]\}.$$

If the second member of (2) is zero, as will be the case if the Q-forces differ from the P-forces only by the impulsive actions and reactions due to the introduction of additional constraints which have no virtual moment in the hypothetical motion into the original system, we have *Bertrand's Theorem:*

If a system in motion is acted on by impulsive forces, the kinetic energy of the subsequent motion is greater than it would be if the system were subjected to any additional constraints and acted on

*by the set of impulsive forces by an amount equal to the energy it would have in the motion which, compounded with the first motion, would give the actual.**

32. By the aid of Thomson's Theorem many problems involving impulsive forces can be treated as simple questions in *maxima and minima*.

(*a*) If, for example, in Art. 28, (*a*), instead of giving the force P we give the velocity v of the foot of the perpendicular from the center of gravity upon the line of the force, so that $\dot{y} + a\dot{\theta} = v$, then to find the motion we have only to make the *energy* $T_{\dot{q}} = \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + k^2\dot{\theta}^2]$ a minimum.

$$\dot{\theta} = \frac{v - \dot{y}}{a}.$$

$$T_{\dot{q}} = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \frac{k^2}{a^2}(v - \dot{y})^2\right].$$

$$\frac{\partial T_{\dot{q}}}{\partial \dot{x}} = m\dot{x} = 0,$$

$$\frac{\partial T_{\dot{q}}}{\partial \dot{y}} = m\left[\dot{y} - \frac{k^2}{a^2}(v - \dot{y})\right] = 0.$$

Whence

$$\dot{x} = 0,$$

$$\dot{y} = \frac{k^2}{a^2 + k^2} v,$$

$$\dot{\theta} = \frac{a}{a^2 + k^2} v,$$

and these results agree entirely with the results obtained in Art. 28, (*a*).

* *Gauss's Principle of Least Constraint*: If a constrained system is acted on by impulsive forces, Gauss takes as the measure of the "constraint" what is practically the kinetic energy of the motion which, combined with the motion that the system would take if all the constraints were removed, would give the actual motion.

It follows easily from Bertrand's Theorem that this "constraint" is less than in any hypothetical motion brought about by introducing additional constraining forces (v. Routh, Elementary Rigid Dynamics, §§ 391–393).

(*b*) In Art. 28, (*c*), since $\dot{x} = v$, the energy

$$T_{\dot{q}} = \frac{m}{2}[8\,v^2 + 3\,\dot{y}^2 - 2\,v\dot{y}].$$

To make $T_{\dot{q}}$ a minimum, we have

$$\frac{dT_{\dot{q}}}{d\dot{y}} = m[3\,\dot{y} - v] = 0,$$

$$\dot{y} = \frac{v}{3},$$

and the problem is solved.

(*c*) In Art. 28, (*d*), let $v_1 = \dot{x} + a\dot{\phi}$ be given.

Then $$T_{\dot{q}} = \frac{m}{2}[4(v_1 - a\dot{\phi})^2 + 4\,\dot{y}^2 + 2(a^2 + k^2)(\dot{\theta}^2 + \dot{\phi}^2)].$$

$$\frac{\partial T_{\dot{q}}}{\partial \dot{y}} = 4\,m\dot{y} = 0,$$

$$\frac{\partial T_{\dot{q}}}{\partial \dot{\theta}} = 2\,m(a^2 + k^2)\dot{\theta} = 0,$$

$$\frac{\partial T_{\dot{q}}}{\partial \dot{\phi}} = m[2(a^2 + k^2)\dot{\phi} - 4\,a(v_1 - a\dot{\phi})] = 0.$$

$$\dot{y} = 0,$$

$$\dot{\theta} = 0,$$

$$\dot{\phi} = \frac{2\,av_1}{3\,a^2 + k^2}.$$

$$\dot{x} = \frac{a^2 + k^2}{3\,a^2 + k^2}v_1 = v_2 = v_4 = \frac{2}{5}v_1,$$

and $$v_3 = \dot{x} - a\dot{\phi} = \frac{-a^2 + k^2}{3\,a^2 + k^2}v_1 = -\frac{1}{5}v_1.$$

33. In using Thomson's Theorem we may employ any valid form in the expression for the energy communicated by the impulsive forces. For instance, in the case of any rigid body,

$$T = \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + A\omega_1^2 + B\omega_2^2 + C\omega_3^2]$$

is permissible and is much simpler than the corresponding form in terms of Euler's coördinates.

Take, for example, the following problem: An elliptic disk is at rest. Suddenly one extremity of the major axis and one extremity of the minor axis are made to move with velocities U and V perpendicular to the plane of the disk. Find the motion of the disk.

Let us take the major axis as the axis of X and the minor axis as the axis of Y.

We have, then, $T = \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + A\omega_1^2 + B\omega_2^2 + C\omega_3^2]$.

By the conditions of the problem, since the components of the velocity of the point $(a, 0, 0)$ are $0, 0, U$, and those of the point $(0, b, 0)$ are $0, 0, V$, we have

$$\dot{x} = 0,$$
$$\dot{y} + a\omega_3 = 0,$$
$$\dot{z} - a\omega_2 = U;$$

and

$$\dot{x} - b\omega_3 = 0,$$
$$\dot{y} = 0,$$
$$\dot{z} + b\omega_2 = V.$$

Hence $$T = \frac{m}{2}\left[\dot{z}^2 + \frac{A}{b^2}(V - \dot{z})^2 + \frac{B}{a^2}(U - \dot{z})^2\right],$$

or $$T = \frac{m}{2}\left[\dot{z}^2 + \frac{1}{4}(V - \dot{z})^2 + \frac{1}{4}(U - \dot{z})^2\right],$$

since $$A = \frac{mb^2}{4} \quad \text{and} \quad B = \frac{ma^2}{4}.$$

$$\frac{dT}{d\dot{z}} = m\left[\dot{z} - \frac{1}{4}(V - \dot{z}) - \frac{1}{4}(U - \dot{z})\right] = 0,$$

$$\dot{z} = \tfrac{1}{6}(U + V).$$

We have also $$\omega_1 = \frac{1}{6b}(5V - U),$$

$$\omega_2 = \frac{1}{6a}(V - 5U),$$

$$\omega_3 = 0.$$

EXAMPLE

One extremity of a side of a square lamina is suddenly made to move perpendicular to the plane of the lamina with velocity U, while the other extremity is made to move in the plane of the lamina and perpendicular to the side with velocity V. Show that the center will move with velocity $\frac{U}{8}$ perpendicular to the plane, and with velocity $\frac{V}{2}\sqrt{2}$ in the plane, toward the corner on which the velocity V was impressed.

34. If our system does not start from rest, it is often easy to frame and to solve a problem in which the system is initially at rest and is acted on by the same impulsive forces as in the actual problem, and where consequently the resulting motion, compounded with the actual initial motion, will give the actual final motion.

For example, consider the following problem: A sphere rotating about any axis is gently placed on a perfectly rough horizontal plane. Find the initial motion. Here, in the actual case, the lowest point of the sphere is immediately reduced to rest.

Take rectangular axes of X and Y, parallel to the plane and through the center of the sphere. Let Ω_x, Ω_y, Ω_z, be the component angular velocities before, and ω_x, ω_y, ω_z after, the sphere is placed on the plane. Let $\dot{x}$, $\dot{y}$, be the velocities of the center of the sphere.

Then, in the actual case, $\dot{x} - a\omega_y = 0$ and $\dot{y} + a\omega_x = 0$ are our given conditions. Initially the velocities of the lowest point of the sphere are $-a\Omega_y$ and $a\Omega_x$. If the sphere were at rest, the impulsive force which in the actual case destroys these velocities would give to the lowest point the negatives of these velocities; that is, $a\Omega_y$ and $-a\Omega_x$. Let us then solve the following auxiliary problem: A sphere is at rest. Suddenly the lowest point is made to move with velocities $a\Omega_y$ and $-a\Omega_x$, parallel to a pair of horizontal axes. Find the initial motion of the sphere.

If u and v are, respectively, the x component and the y component of the velocity of the center, and ω_1, ω_2, ω_3, are the angular velocities,

$$T=\frac{m}{2}\left[u^2+v^2+k^2(\omega_1^2+\omega_2^2+\omega_3^2)\right].$$

The velocities of the lowest point are $u-a\omega_2$, $v+a\omega_1$, but they were given as $a\Omega_y$ and $-a\Omega_x$.

Therefore
$$u-a\omega_2=a\Omega_y,$$
$$v+a\omega_1=-a\Omega_x.$$
$$T=\frac{m}{2}\{a^2[(\omega_2+\Omega_y)^2+(\omega_1+\Omega_x)^2+k^2(\omega_1^2+\omega_2^2+\omega_3^2)]\}.$$

To make this a minimum, we have

$$\frac{\partial T}{\partial\omega_1}=m\left[a^2(\omega_1+\Omega_x)+k^2\omega_1\right]=0,$$
$$\frac{\partial T}{\partial\omega_2}=m\left[a^2(\omega_2+\Omega_y)+k^2\omega_2\right]=0,$$
$$\frac{\partial T}{\partial\omega_3}=mk^2\omega_3=0.$$

Hence
$$\omega_1=-\frac{a^2}{a^2+k^2}\Omega_x,$$
$$\omega_2=-\frac{a^2}{a^2+k^2}\Omega_y,$$
$$\omega_3=0.$$

Compounding these with the initial angular velocities in the actual problem, Ω_x, Ω_y, Ω_z, we get

$$\omega_x=\frac{k^2}{a^2+k^2}\Omega_x,$$
$$\omega_y=\frac{k^2}{a^2+k^2}\Omega_y,$$
$$\omega_z=\Omega_z.$$

These equations, together with $\dot{x}-a\omega_y=0$ and $\dot{y}+a\omega_x=0$, completely solve the original problem.

35. A Problem in Fluid Motion. Let us now consider an interesting application of the principles of this chapter which was made by Lord Kelvin to a problem in fluid motion.

It is shown in treatises on hydromechanics that if an incompressible, frictionless, homogeneous liquid, either infinite in extent or bounded by any finite closed surfaces fixed or moving, and with any rigid or flexible bodies immersed in it, is moving under the action of *conservative* forces (v. Chap. IV) and has ever been at rest, the motion will be what is called *irrotational.* That is, if x, y, z, are the rectangular coördinates of any fixed point in the space occupied by the liquid, there will be a function $\phi(x, y, z)$ such that if u, v, w, are the components of the velocity of the liquid at the point x, y, z,

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z}.$$

The function ϕ is called the *velocity-potential* function.

Since throughout the motion the liquid is always supposed to be an *incompressible continuum*, u, v, w, must satisfy the equation of continuity for an incompressible liquid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0;$$

and therefore ϕ satisfies Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

and will be uniquely determined except for an arbitrary constant term if the value of $\frac{\partial \phi}{\partial n}$, the velocity normal to the surface, is given at every point of the boundary of the liquid, however irregular that boundary. Therefore the actual motion at every point of the liquid at any instant is uniquely determined, if the motion is irrotational, when the normal velocities at all points of the boundary are given.

We wish now to prove that the kinetic energy of the actual motion is less than that of any other motion, not necessarily

irrotational, consistent with the equation of continuity and with the actual normal velocities at the boundary.

If u, v, w, are the velocities at (x, y, z) in the actual motion, let $u+\alpha$, $v+\beta$, $w+\gamma$, be the velocities in the hypothetical motion, and let v_n be the actual normal velocity at any point of the boundary. Then we have $\frac{\partial\phi}{\partial n} = lu + mv + nw = v_n$, where l, m, n, are the direction cosines of the normal, and u, v, w, are the components of the velocity at the point in question.

In the hypothetical motion,

$$l(u+\alpha) + m(v+\beta) + n(w+\gamma) = v_n$$

at the same point; therefore

$$l\alpha + m\beta + n\gamma = 0 \tag{1}$$

at every point of the boundary.

As the hypothetical velocities as well as the actual velocities must obey the law of continuity,

$$\frac{\partial(u+\alpha)}{\partial x} + \frac{\partial(v+\beta)}{\partial y} + \frac{\partial(w+\gamma)}{\partial z} = 0,$$

and therefore
$$\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z} = 0 \tag{2}$$

at every point in the bounded space.

If T is the energy of the actual motion,

$$T = \frac{\rho}{2}\iiint [u^2 + v^2 + w^2]\, dx\, dy\, dz,$$

where ρ is the density of the liquid, and where the volume integral is taken throughout the space filled by the liquid.

If T' is the energy of the hypothetical motion,

$$\begin{aligned} T' &= \frac{\rho}{2}\iiint [(u+\alpha)^2 + (v+\beta)^2 + (w+\gamma)^2]\, dx\, dy\, dz \\ &= T + \frac{\rho}{2}\iiint [\alpha^2 + \beta^2 + \gamma^2]\, dx\, dy\, dz \\ &\quad + \rho\iiint [\alpha u + \beta v + \gamma w]\, dx\, dy\, dz. \end{aligned}$$

By the aid of Green's Theorem we can prove that

$$\iiint [\alpha u + \beta v + \gamma w]\, dx\, dy\, dz = 0.$$

We have $$\iiint \frac{\partial U}{\partial x}\, dx\, dy\, dz = \int U \cos \alpha\, dS$$

(v. Peirce, Newtonian Potential Function, p. 92 (143)), where the volume integral is taken throughout any bounded space and the surface integral over the boundary of the space, $\cos \alpha$ being the x direction cosine of the normal to the boundary.

Now $$\alpha u = \alpha \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(\alpha\phi) - \phi \frac{\partial \alpha}{\partial x};$$

and $$\iiint \alpha u\, dx\, dy\, dz = \int l\alpha\phi dS - \iiint \phi \frac{\partial \alpha}{\partial x}\, dx\, dy\, dz.$$

In like manner,

$$\iiint \beta v\, dx\, dy\, dz = \int m\beta\phi dS - \iiint \phi \frac{\partial \beta}{\partial y}\, dx\, dy\, dz,$$

and $$\iiint \gamma w\, dx\, dy\, dz = \int n\gamma\phi dS - \iiint \phi \frac{\partial \gamma}{\partial z}\, dx\, dy\, dz.$$

Hence $$\iiint [\alpha u + \beta v + \gamma w]\, dx\, dy\, dz$$

$$= \int [l\alpha + m\beta + n\gamma]\phi dS - \iiint \phi \left[\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}\right] dx\, dy\, dz.$$

But the surface integral vanishes by (1), and the volume integral vanishes by (2). Therefore the energy T' is greater than the actual energy T.

It follows that the irrotational motion of any frictionless incompressible homogeneous liquid under the action of conservative forces is at every instant identical with the motion which would have been suddenly generated from rest by a set of impulsive forces applied at points in the boundary of the liquid and such that they would suddenly give all the points of the boundary the normal velocities that these points actually have at the instant in question (v. Thomson's Theorem in Art. 30).

36. If now we have a liquid contained in a material vessel and containing immersed bodies, a set of generalized coördinates $q_1, q_2, \cdots, q_n$, can be chosen, equal to the number of degrees of freedom of the material system formed by the vessel and the immersed bodies, and the normal velocity of every point of the surface of the vessel and of the surfaces of the immersed bodies can be expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$, and the corresponding generalized velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$.

We can now choose other independent coördinates $q_1', q_2', \cdots$, practically infinite in number, which, together with our coördinates $q_1, q_2, \cdots, q_n$, will give the positions of all the particles of the liquid.

Suppose the system (vessel, immersed bodies, and liquid) at rest. Apply any set of impulsive forces, not greater in number than n, at points in the surface of vessel and of immersed bodies and consider the equations of motion. For any of our coördinates q_k' we have the equation

$$p_k' = 0,$$

where p_k' is the generalized momentum corresponding to q_k', since (as in varying q_k' no one of the coördinates $q_1, q_2, \cdots, q_n$, is changed, and therefore no one of the impulsive forces has its point of application moved) the virtual moment of the component impulse corresponding to q_k' is zero.

As the actual motion at every instant could have been generated suddenly from rest by such impulsive forces as we have just considered, the momentum p_k' is zero throughout the actual motion; and the impressed forces being by hypothesis conservative, and the liquid always forming a *continuum*, no work is done when q_k' is varied. Consequently, in the Hamiltonian equation $\dot{p}_k' + \frac{\partial T_p}{\partial q_k'} = Q_k'$, $\dot{p}_k' = 0$ and $Q_k' = 0$, and therefore $\frac{\partial T_p}{\partial q_k'} = 0$. Hence every coördinate q_k' is *cyclic*, and it is also completely ignorable. The energy *modified* for the coördinates q_k' is identical with the energy, which, being free from the coördinates q_k'

and the momenta p_k', is expressible in terms of the n coördinates $q_1, q_2, \cdots, q_n$, and the corresponding velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, and is a homogeneous quadratic in terms of these velocities (v. Art. 24, (*a*)).

37. Summary of Chapter III. In dealing with problems in which a moving system is supposed to be acted on by impulsive forces, we care only for the *state of motion* brought about by the forces in question, since on the usual assumption that there is no change in configuration during the action of the impulsive forces, we are not concerned with the values of the coördinates but merely with the values of their time derivatives.

The virtual moment (v. Art. 26) of the effective impulsive forces in a hypothetical infinitesimal displacement of the system, due to an infinitesimal change δq_k in a single coördinate q_k, is

$$\left\{\left[\frac{\partial T_{\dot{q}}}{\partial \dot{q}_k}\right]_1 - \left[\frac{\partial T_{\dot{q}}}{\partial \dot{q}_k}\right]_0\right\}\delta q_k \quad \text{or} \quad [(p_k)_1 - (p_k)_0]\,\delta q_k.$$

If either of these is written equal to $P_k \delta q_k$, the virtual moment of the actual impulsive forces in the displacement in question, we have one of the equivalent equations

$$\left[\frac{\partial T_{\dot{q}}}{\partial \dot{q}_k}\right]_1 - \left[\frac{\partial T_{\dot{q}}}{\partial \dot{q}_k}\right]_0 = P_k,$$

$$(p_k)_1 - (p_k)_0 = P_k.$$

The n equations of which either of these is the type are n simultaneous linear equations in the n final velocities $(\dot{q}_1)_1$, $(\dot{q}_2)_1$, $\cdots$, and as the configuration and the initial state of motion are supposed to be given, no integration is required, and the problem becomes one in elementary algebra.

A skillful use of Thomson's or of Bertrand's Theorem reduces many problems in motion under impulsive forces to simple problems in *maxima and mimima.*

CHAPTER IV

CONSERVATIVE FORCES

38. If X, Y, Z, are the components of the forces acting on a moving particle (coördinates x, y, z), the work, W, done by the forces while the particle moves from a given position P_0, (x_0, y_0, z_0), to a second position P_1, (x_1, y_1, z_1), is equal to

$$\int_{P_0}^{P_1}[Xdx + Ydy + Zdz];$$

and since every one of the quantities X, Y, and Z is in the general case a function of the three variables x, y, z, we need to know the path followed by the moving particle, in order to find W. Let $f(x, y, z) = 0$, $\phi(x, y, z) = 0$, be the equations of the path. We can eliminate z between these equations and then express y explicitly in terms of x; we can then eliminate y between the same two equations and express z in terms of x; and we can substitute these values for y and z in X, which will then be a function of the single variable x, and $\int_{x_0}^{x_1} Xdx$ can be found by a simple quadrature. By proceeding in the same way with Y and Z we can find $\int_{y_0}^{y_1} Ydy$ and $\int_{z_0}^{z_1} Zdz$, and the sum of these three integrals will be the work required.

It may happen, however, that $Xdx + Ydy + Zdz$ is what is called an *exact differential*, that is, that there is a function $U = \phi(x, y, z)$ such that

$$\frac{\partial U}{\partial x} = X, \qquad \frac{\partial U}{\partial y} = Y, \qquad \frac{\partial U}{\partial z} = Z.$$

Since the complete differential of this function is

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz,$$

or $Xdx + Ydy + Zdz$, we have

$$\int [Xdx + Ydy + Zdz] = \phi(x, y, z) = U,$$

and
$$\int_{P_0}^{P_1} [Xdx + Ydy + Zdz]$$
$$= \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) = U_1 - U_0;$$

and in obtaining this result we have made no use of the path followed by the moving particle.

When the forces are such that the function $U = \phi(x, y, z)$ exists, they are said to be *conservative*, and U is called the *force function.*

We can infer, then, that the work done by conservative forces on a particle moving by any path from a given initial position to a given final position is independent of the path and is equal to the value of the force function in the final position minus its value in the initial position.

If instead of a moving particle we have a system of particles, the reasoning given above applies.

Let (x_k, y_k, z_k) be any particle of the system, and X_k, Y_k, Z_k, be the forces applied at the particle. Then the whole work, W, done on the system as it moves from one configuration to another is equal to

$$\sum \int_{(P_k)_0}^{(P_k)_1} [X_k dx_k + Y_k dy_k + Z_k dz_k].$$

If there is a function $U = \phi(x_1, x_2, \cdots, x_k, \cdots, y_1, y_2, \cdots, y_k, \cdots, z_1, z_2, \cdots, z_k, \cdots)$ such that

$$\frac{\partial U}{\partial x_k} = X_k, \qquad \frac{\partial U}{\partial y_k} = Y_k, \qquad \frac{\partial U}{\partial z_k} = Z_k,$$

then $\Sigma\,[X_k dx_k + Y_k dy_k + Z_k dz_k]$ is an *exact differential* and $U = \phi(x_1, x_2, \cdots, y_1, y_2, \cdots, z_1, z_2, \cdots)$ is its indefinite integral and is the *force function*; the forces are a *conservative* set; and

the work done by the forces as the system moves from a given initial to a given final configuration is equal to the value of the force function in the final configuration minus its value in the initial configuration, no matter by what paths the particles may have moved from their initial to their final positions.

It is well known and can be shown without difficulty that such forces as gravity, the attraction of gravitation, any mutual attraction or repulsion between particles of a system which for every pair of particles acts in the line joining the particles and is a function of their distance apart, are conservative; while such forces as friction, or the resistance of the air or of a liquid to the motion of a set of particles, are not conservative.

The negative of the force function of a system moving under conservative forces is called the *potential energy* of the system, and we shall represent it by V.

If we are dealing with motion under conservative forces and are using generalized coördinates, and the geometrical equations do not contain the time, we can replace the rectangular coördinates of the separate particles of the system in the force function or in the potential energy by their values in terms of the generalized coördinates $q_1, q_2, \cdots, q_n$, and we can thus get U, and consequently V, expressed in terms of the generalized coördinates.

If U is thus expressed, $Q_k\delta q_k$ (the work done by the impressed forces when the system is displaced by changing q_k by δq_k) is $\delta_{q_k}U$ and therefore is approximately $\frac{\partial U}{\partial q_k}\delta q_k$, or $-\frac{\partial V}{\partial q_k}\delta q_k$, and hence

$$Q_k = \frac{\partial U}{\partial q_k} = -\frac{\partial V}{\partial q_k}.$$

39. The Lagrangian and the Hamiltonian Functions. If the forces are conservative, our Lagrangian equation

$$\frac{d}{dt}\frac{\partial T_{\dot q}}{\partial \dot q_k} - \frac{\partial T_{\dot q}}{\partial q_k} = Q_k \tag{1}$$

may be written

$$\frac{d}{dt}\frac{\partial T_{\dot q}}{\partial \dot q_k} = \frac{\partial T_{\dot q}}{\partial q_k} - \frac{\partial V}{\partial q_k}, \tag{2}$$

where V is the potential energy expressed in terms of the coördinates $q_1, q_2, \cdots$, and not containing the velocities $\dot{q}_1, \dot{q}_2, \cdots$.

If
$$L = T_{\dot{q}} - V, \tag{3}$$
L is an explicit function of the coördinates and the velocities and is called the *Lagrangian function.*

Obviously $\dfrac{\partial L}{\partial \dot{q}_k} = \dfrac{\partial T_{\dot{q}}}{\partial \dot{q}_k}$, and $\dfrac{\partial L}{\partial q_k} = \dfrac{\partial T_{\dot{q}}}{\partial q_k} - \dfrac{\partial V}{\partial q_k}$.

Hence our Lagrangian equation (1) can be written very neatly
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}. \tag{4}$$

If our forces are conservative, and we are using the Hamiltonian equations and express the kinetic energy T_p in terms of the coördinates and the corresponding momenta, and if we let
$$H = T_p + V, \tag{5}$$
H is called the *Hamiltonian function* and is a function of $q_1, q_2, \cdots, q_n$, and $p_1, p_2, \cdots, p_n$.

Our Hamiltonian equations
$$\dot{p}_k + \frac{\partial T_p}{\partial q_k} = Q_k,$$
$$\dot{q}_k = \frac{\partial T_p}{\partial p_k},$$
can now be written
$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \tag{6}$$
$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \tag{7}$$
and these are known as the *Hamiltonian canonical equations.*

If our forces are conservative, and we are using instead of the kinetic energy the Lagrangian expression for the energy modified for some of the coördinates $q_1, q_2, \cdots, q_r$ (v. Art. 17), and if we let
$$\Phi = M_{q_1, \cdots, q_r} - V, \tag{8}$$

we have for any coördinate q_{r+k} the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{q}_{r+k}}\right)=\frac{\partial \Phi}{\partial q_{r+k}}, \tag{9}$$

and for any coördinate q_{r-k} the pair of Hamiltonian equations

$$\frac{dp_{r-k}}{dt}=\frac{\partial \Phi}{\partial q_{r-k}}, \tag{10}$$

$$\frac{dq_{r-k}}{dt}=-\frac{\partial \Phi}{\partial p_{r-k}}. \tag{11}$$

40. The Lagrangian function L is the difference between the kinetic energy $T_{\dot{q}}$ (expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$, and the velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$, and homogeneous of the second degree in terms of the velocities) and the potential energy V (expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$).

The Hamiltonian function H is the sum of the kinetic energy T_p (expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$, and the corresponding momenta $p_1, p_2, \cdots, p_n$, and homogeneous of the second degree in terms of the momenta) and the potential energy V (expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$).

The sum of the kinetic energy and the potential energy, however expressed, is sometimes called the *total energy* of the system, and we shall represent it by E, so that

$$E=T+V. \tag{1}$$

The function Φ of the preceding section is the difference between the kinetic energy (expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$, the momenta $p_1, p_2, \cdots, p_r$, and the velocities $\dot{q}_{r+1}, \dot{q}_{r+2}, \cdots, \dot{q}_n$), minus the terms

$$p_1\dot{q}_1+p_2\dot{q}_2+\cdots+p_r\dot{q}_r$$

(similarly expressed), and the potential energy (expressed in terms of the coördinates $q_1, q_2, \cdots, q_n$). We shall call it the *modified Lagrangian function.*

It is to be observed that all the terms of Φ except those contributed by the potential energy are homogeneous of the second degree in the momenta introduced and the velocities not eliminated by the modification.

41. In dealing with the motion of a system under conservative forces, we may form the differential equations of motion in any one of three ways, and the equations in question are practically given by giving a single function — L, the Lagrangian function, or H, the Hamiltonian function, or Φ, the modified Lagrangian function.*

Every one of these functions consists of two very different parts: one, the potential energy V, which depends merely on the forces, which in turn depend solely on the configuration of the system; the other, the kinetic energy T or the modified Lagrangian expression $M_{q_1},\ldots$, either of which involves the velocities or the momenta of the system as well as its configuration.

If we are using as many independent coördinates as there are degrees of freedom, a mere inspection of the given function will enable us to distinguish between the two functions of which it is formed, the potential energy or its negative being composed of all the terms not involving velocities or momenta.

If, however, we are ignoring some of the coördinates (v. Arts. 16, 21, and 24) and are using H (the Hamiltonian function) or Φ (the Lagrangian function modified for the ignored coördinates), the portion contributed to H by T_p or to Φ by M (the modified expression for the kinetic energy) is no longer necessarily a homogeneous quadratic in the velocities and momenta (v. Art. 21) and may contain terms involving merely the coördinates and therefore indistinguishable from terms belonging to the potential energy; and consequently the part of the motion not ignored would be identical with that which would be produced by a set of forces quite different from the actual forces.

* Indeed, for equations of the Lagrangian type, any constant multiple of L or Φ will serve as well as L or Φ.

We may note that the last paragraph does not apply if the system starts from rest, so that the ignored momenta are zero throughout the motion (v. Art. 24, (*a*)).

Let us consider the problem of Art. 8, (*d*), where the potential energy V is easily seen to be $-mgx$.

If we are using the Lagrangian method, as in Art. 8, (*d*), we have

$$L = \frac{m}{2}\left[2\dot{x}^2 + (a-x)\dot{\theta}^2 + 2gx\right]. \tag{1}$$

If we use the Hamiltonian method, as in Art. 15, (*d*), we have

$$H = \frac{1}{2m}\left[\frac{p_x^2}{2} + \frac{p_\theta^2}{(a-x)^2} - 2m^2gx\right]. \tag{2}$$

If we use the Lagrangian method modified for θ, as in Art. 20, (*b*), we have

$$\Phi = \frac{m}{2}\left[2\dot{x}^2 - \frac{p_\theta^2}{m^2(a-x)^2} + 2gx\right]. \tag{3}$$

A mere inspection of any one of these three functions enables us to pick out the potential energy as $V = -mgx$.

If, however, we are ignoring θ, as in Art. 22, Example, we have

$$\Phi = \frac{m}{2}\left[2\dot{x}^2 - \frac{c^2}{m^2(a-x)^2} + 2gx\right], \tag{4}$$

and here, so far as the function Φ shows, the potential energy may be $-mgx$, as, in fact, it is, or it may be $\dfrac{c^2}{2m(a-x)^2} - mgx$. Indeed, the hanging particle moves precisely as if its mass were $2m$ and it were acted on by a force having the force function

$$U = -\frac{c^2}{2m(a-x)^2} + mgx;$$

that is, a force vertically downward equal to $mg - \dfrac{c^2}{m(a-x)^3}$.

If, as in many important problems (v. Chap. V), we are unable to discern and measure the impressed forces directly and are attempting to deduce them from observations on the behavior of

a complicated system, which for aught we know may contain undetected moving masses, the fact that we cannot discriminate with certainty between the terms contributed to the modified Lagrangian function by the kinetic energy of the system and those contributed by the potential energy may lead to entirely different and equally plausible explanations of the observed phenomena (v. Art. 51).

42. Conservation of Energy. If we are dealing with a system moving under conservative forces, the coördinates $q_1, q_2, \cdots, q_n$, are functions of t, the time, as are also the velocities $\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n$. Therefore V, the potential energy, and T, the kinetic energy, are functions of the time, as is their sum, the Hamiltonian function H.

Let us find $\dfrac{dH}{dt}$.

As H depends explicitly on the coördinates $q_1, q_2, \cdots, q_n$, and the momenta $p_1, p_2, \cdots, p_n$,

$$\frac{dH}{dt} = \sum \left[\frac{\partial H}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial H}{\partial p_k} \frac{dp_k}{dt} \right]. \tag{1}$$

But by our Hamiltonian equations (v. Art. 39, (6) and (7)),

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \quad \text{and} \quad \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k},$$

and (1) becomes

$$\frac{dH}{dt} = \sum \left[\frac{\partial H}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial H}{\partial q_k} \right] = 0. \tag{2}$$

Therefore $$T + V = H = h, \tag{3}$$
where h is a constant.

Hence *in any system moving under conservative forces the sum of the kinetic energy and the potential energy is constant during the motion.*

This is called the Principle of the Conservation of Energy.

Since by (3) any loss in potential energy during the motion is just balanced by an increase in the kinetic energy, and the loss in potential energy is equal to the work done by the actual

forces during the motion, our principle is a narrower statement of the familiar principle: *If a system is moving under any forces, conservative or not, the gain in kinetic energy is always equal to the work done by the actual forces.*

43. Hamilton's Principle. Let a system move under conservative forces from its configuration at the time t_0 to its configuration at the time t_1. We have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}, \tag{1}$$

where L, the Lagrangian function, is equal to $T - V$.

Suppose that the system had been made to move from the first to the second configuration so that the particles traced slightly different paths with slightly different velocities, but so that at any time t every coördinate q_k differed from its value in the actual motion by an infinitesimal amount, and so that every velocity $\dot{q}_k$ differed from its value in the actual motion by an infinitesimal amount, or (using the notation of the *calculus of variations*)* so that δq_k and $\delta \dot{q}_k$ were infinitesimal; and suppose it had reached the second configuration at the time t_1. Then, if at the time t the difference between the value of L in the hypothetical motion and its value in the actual motion is δL,

$$\delta L = \sum \left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right]. \tag{2}$$

Now, at the time t,

$$\delta \dot{q}_k = \frac{d}{dt}\delta q_k,$$

and

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k &= \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\delta q_k \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\delta q_k \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) - \frac{\partial L}{\partial q_k}\delta q_k, \text{ by (1).}\end{aligned}$$

* For a brief introduction to the calculus of variations, see Appendix B.

Therefore, by (2), $\delta L = \frac{d}{dt}\sum\left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]$,

and $$\int_{t_0}^{t_1}\delta L dt = \delta\int_{t_0}^{t_1} L dt = \left[\sum\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_0}^{t_1}.$$

Since the terminal configurations are the same in the actual motion and in the hypothetical motion and the time of transit is the same, $\delta q_k = 0$ when $t = t_0$ and when $t = t_1$, and

$$\delta\int_{t_0}^{t_1} L dt = \delta\int_{t_0}^{t_1}[T - V]\,dt = 0. \qquad (3)$$

But (3) is the necessary condition that L should be either a *maximum* or a *minimum* and is sometimes stated as follows: *When a system is moving under conservative forces, the time integral of the difference between the kinetic energy and the potential energy of the system is "stationary" in the actual motion.* This is known as *Hamilton's principle.**

44. The Principle of Least Action. If the limitation in the preceding section that "in the actual and the hypothetical motions the time of transit from the first to the second configuration is

* Hamilton's principle plays so important a part in mechanics and physics that it seems worth while to obtain a formula for it which is not restricted to conservative systems.

We shall use rectangular coördinates, and we shall make the hypothesis as to the actual motion and the hypothetical motion which has been employed above.

For every particle of the system we have the familiar equations

$$m\ddot{x} = X, \qquad m\ddot{y} = Y, \qquad m\ddot{z} = Z.$$

Since $$T = \sum\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

$$\delta T = \Sigma m[\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z}] = \Sigma m\left[\dot{x}\frac{d}{dt}\delta x + \dot{y}\frac{d}{dt}\delta y + \dot{z}\frac{d}{dt}\delta z\right].$$

But $$m\dot{x}\frac{d}{dt}\delta x = m\frac{d}{dt}(\dot{x}\delta x) - m\ddot{x}\delta x = \frac{d}{dt}(m\dot{x}\delta x) - X\delta x.$$

Hence $$\delta T + \Sigma[X\delta x + Y\delta y + Z\delta z] = \frac{d}{dt}\Sigma m[\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z],$$

and $$\int_{t_0}^{t_1}\{\delta T + \Sigma[X\delta x + Y\delta y + Z\delta z]\}\,dt = \Sigma[m\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z]_{t_0}^{t_1} = 0.$$

If the system is conservative, $\Sigma[X\delta x + Y\delta y + Z\delta z] = \delta U = -\delta V$, and we get the formula (3) in the text.

the same" be removed and the variations be taken not at the same time but at arbitrarily corresponding times, t will no longer be the independent variable but will be regarded as depending upon some independent parameter r. We now have (v. Art. 43)

$$\delta \dot{q}_k = \frac{d}{dt}\delta q_k - \dot{q}_k \frac{d}{dt}\delta t, \qquad \text{(v. App. B, § 6, (1))}$$

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k &= \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\delta q_k - \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\delta t \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) - \frac{\partial L}{\partial q_k}\delta q_k - \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\delta t,\end{aligned}$$

and

$$\begin{aligned}\delta L &= \frac{d}{dt}\sum\left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right] - \frac{d}{dt}\delta t \cdot \sum\left[\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right] \\ &= \frac{d}{dt}\sum\left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right] - 2\,T\frac{d}{dt}\delta t,\end{aligned}$$

since

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T_{\dot{q}}}{\partial \dot{q}_k}.$$

Hence
$$\delta L + 2\,T\frac{d}{dt}\delta t = \frac{d}{dt}\sum\left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right],$$

and
$$\begin{aligned}\int_{r_0}^{r_1}\left[\delta L + 2\,T\frac{d}{dt}\delta t\right]\frac{dt}{dr}dr &= \int_{r_0}^{r_1}\frac{d}{dt}\left[\sum \frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]\frac{dt}{dr}dr \\ &= \left[\sum \frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{r_0}^{r_1} = 0. \qquad (1)\end{aligned}$$

If now we impose the condition that during the hypothetical motion, as during the actual motion, the equation of the conservation of energy holds good, that is, that

$$T + V = h,$$

then
$$\delta T + \delta V = 0,$$

$$\delta L = \delta T - \delta V = 2\,\delta T = \delta(2\,T),$$

and (1) becomes

$$\int_{r_0}^{r_1}\left[\delta(2\,T) + 2\,T\frac{d}{dt}\delta t\right]\frac{dt}{dr}dr = 0,$$

$$\int_{r_0}^{r_1}\left[\delta(2\,T)\frac{dt}{dr}+2\,T\frac{d}{dr}\delta t\right]dr=0,$$

$$\int_{r_0}^{r_1}\left[\frac{dt}{dr}\delta(2\,T)+2\,T\delta\frac{dt}{dr}\right]dr=0,$$

$$\int_{r_0}^{r_1}\delta\left(2\,T\frac{dt}{dr}\right)dr=0,$$

$$\delta\int_{r_0}^{r_1}2\,T\frac{dt}{dr}\,dr=0,$$

and, finally,
$$\delta\int_{t_0}^{t_1}2\,Tdt=0. \tag{2}$$

The equation $A=\int_{t_0}^{t_1}2\,T\,dt$ defines the *action*, A, of the forces, and the fact stated in (2) is usually called the *principle of least action*. As a matter of fact, (2) shows merely that the *action* is "stationary."

45. In establishing Hamilton's principle we have supposed the *course* followed by the system to be varied, subject merely to the limitation that the time of transit from initial to final configuration should be unaltered; consequently, as the *total energy* is not conserved, the varied course is not a *natural* course. That is, to compel the system to follow such a course we should have to introduce additional forces that would do work.

In establishing the principle of least action, however, we have supposed the *course* followed by the system to be varied, subject to the limitation that the total energy should be unaltered; consequently the varied course is a *natural* course. That is, to compel the system to follow such a course we need introduce merely suitable constraints that would do no work.

We have deduced both principles from the equations of motion. Conversely, the equations of motion can be deduced from either of them. Each of them is therefore a necessary and sufficient condition for the equations of motion.

46. The action, A, of a conservative set of forces acting on a moving system has been defined as the time integral of twice the kinetic energy.

$$A = \int_{t_0}^{t_1} 2T dt. \tag{1}$$

But $$A = \int_{t_0}^{t_1} \Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt = \int_{t_0}^{t_1} \Sigma m\left(\frac{ds}{dt}\right)^2 dt = \int_{t_0}^{t_1} \Sigma mv^2 dt.$$

Therefore $$A = \int \Sigma mvds, \tag{2}$$

so that the *action* might just as well have been defined as the sum of terms any one of which is the line integral of the *momentum* of a particle taken along the actual path of the particle.

There is another interesting expression for the *action*, which does not involve the time even through a velocity.

Since $$T + V = h, \qquad 2T = 2(h - V).$$

As $$T = \sum \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$$

$$\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 2(h - V),$$

$$\Sigma m\left(\frac{ds}{dt}\right)^2 = 2(h - V),$$

$$\Sigma mds^2 = 2(h - V)dt^2,$$

and $$dt = \sqrt{\frac{\Sigma mds^2}{2(h - V)}}.$$

$$A = \int 2(h - V)\sqrt{\frac{\Sigma mds^2}{2(h - V)}}.$$

$$A = \int \sqrt{2(h - V)\Sigma mds^2}. \tag{3}$$

47. We have stated without proof that the differential equations of motion for any system under conservative forces could be deduced from Hamilton's principle or from the principle of stationary action. Instead of giving the proof in general, we will give it in a concrete case, that of a projectile under gravity.

We shall use fixed rectangular coördinates, Y horizontal and X vertically downward, taking the origin at the starting point of the projectile. In this case, obviously,

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \quad \text{and} \quad V = -mgx.$$

(*a*) By Hamilton's principle, we have

$$\delta \int_0^{t_1} \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + 2gx]\,dt = 0, \tag{1}$$

or

$$\int_0^{t_1} \delta[\dot{x}^2 + \dot{y}^2 + 2gx]\,dt = 0.$$

$$\int_0^{t_1} [\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + g\delta x]\,dt = 0.$$

$$\int_0^{t_1} \left[\dot{x}\frac{d}{dt}\delta x + \dot{y}\frac{d}{dt}\delta y + g\delta x\right]dt = 0.$$

Integrating by parts,

$$\Big[\dot{x}\delta x + \dot{y}\delta y\Big]_0^{t_1} - \int_0^{t_1}\left[\left(\frac{d\dot{x}}{dt} - g\right)\delta x + \frac{d\dot{y}}{dt}\delta y\right]dt = 0,$$

$$\int_0^{t_1}\left[\left(\frac{d^2x}{dt^2} - g\right)\delta x + \frac{d^2y}{dt^2}\delta y\right]dt = 0;$$

but since δx and δy are wholly arbitrary, this is impossible unless the coefficients of δx and δy in the integrand are both zero.

Hence
$$\frac{d^2x}{dt^2} - g = 0,$$
$$\frac{d^2y}{dt^2} = 0.$$

(*b*) By the principle of stationary action, we have

$$\delta \int_{r_0}^{r_1} m(\dot{x}^2 + \dot{y}^2)\frac{dt}{dr}\,dr = 0,$$

whence
$$\int_{r_0}^{r_1} (\dot{x}^2 + \dot{y}^2)\,\delta\frac{dt}{dr}\,dr + \int_{r_0}^{r_1} \frac{dt}{dr}\,\delta(\dot{x}^2 + \dot{y}^2)\,dr = 0. \tag{1}$$

But $$\frac{m}{2}(\dot{x}^2+\dot{y}^2)-mgx=h.$$

Hence $$\delta\frac{(\dot{x}^2+\dot{y}^2)}{2}=g\delta x,$$

and $$\delta(\dot{x}^2+\dot{y}^2)=\delta\frac{\dot{x}^2+\dot{y}^2}{2}+g\delta x.$$

$$\delta\frac{\dot{x}^2+\dot{y}^2}{2}=\dot{x}\delta\dot{x}+\dot{y}\delta\dot{y}=\dot{x}\frac{d}{dt}\delta x+\dot{y}\frac{d}{dt}\delta y-(\dot{x}^2+\dot{y}^2)\frac{d}{dt}\delta t,$$

since $$\delta\dot{x}=\delta\frac{dx}{dt}=\frac{d}{dt}\delta x-\frac{dx}{dt}\frac{d}{dt}\delta t;\quad \text{(v. App. B, § 6, (1))}$$

$$\frac{dt}{dr}\delta\frac{\dot{x}^2+\dot{y}^2}{2}=\dot{x}\frac{d}{dr}\delta x+\dot{y}\frac{d}{dr}\delta y-(\dot{x}^2+\dot{y}^2)\frac{d}{dr}\delta t;$$

and (1) becomes

$$\int_{r_0}^{r_1}(\dot{x}^2+\dot{y}^2)\frac{d}{dr}\delta t\,dr+\int_{r_0}^{r_1}\left[\dot{x}\frac{d}{dr}\delta x+\dot{y}\frac{d}{dr}\delta y\right]dr$$
$$-\int_{r_0}^{r_1}(\dot{x}^2+\dot{y}^2)\frac{d}{dr}\delta t\,dr+\int_{r_0}^{r_1}g\delta x\frac{dt}{dr}dr=0,$$

or $$\int_{r_0}^{r_1}\left[\dot{x}\frac{d}{dr}\delta x+\dot{y}\frac{d}{dr}\delta y\right]dr+\int_{r_0}^{r_1}g\delta x\frac{dt}{dr}dr=0.$$

Integrating by parts, we get

$$\left[\dot{x}\delta x+\dot{y}\delta y\right]_{r_0}^{r_1}-\int_{r_0}^{r_1}\left[\ddot{x}\frac{dt}{dr}\delta x+\ddot{y}\frac{dt}{dr}\delta y-g\frac{dt}{dr}\delta x\right]dr=0.$$

But δx and δy are zero at the beginning and at the end of the actual and the hypothetical paths, so that

$$\int_{r_0}^{r_1}[(\ddot{x}-g)\,\delta x+\ddot{y}\delta y]\frac{dt}{dr}dr=0,$$

and as δx and δy are wholly arbitrary, this necessitates that

$$\ddot{x}=g,$$
$$\ddot{y}=0,$$

as in Art. 47, (a).

48. Although we have proved merely that in a system under conservative forces the action satisfies the necessary condition of a minimum, namely, $\delta A = 0$, it may be shown by an elaborate investigation that in most cases it actually is a minimum, and that the name "least action," usually associated with the principle, is justified.

A very pretty corollary of the principle comes from its application to a system moving under no forces or under constraining forces that do no work. In either of these cases the potential energy V is zero, and consequently the kinetic energy is constant, that is, $T = h$.

$$A = \int_{t_0}^{t_1} 2\, T dt = 2\, h \int_{t_0}^{t_1} dt = 2\, h\,(t_1 - t_0)$$

and the action is proportional to the time of transit. Hence the actual motion is along the *course* which occupies the least possible time.

For instance, if a particle is moving under no forces, the energy $\frac{m}{2} v^2$ is constant, the velocity of the particle is constant, and as it moves from start to finish in the least possible time, it must move from start to finish by the shortest possible path, that is, by a straight line.

If instead of moving freely the particle is constrained to move on a given surface, the same argument proves that it must trace a geodetic on the surface.

49. Varying Action.* The action, A, between two configurations of a system under conservative forces is theoretically expressible in terms of the initial and final coördinates and the total energy (v. Art. 46, (3)), and, when so expressed, Hamilton called it the *characteristic function.*

In like manner, if

$$S = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (T - V)\, dt = \int_{t_0}^{t_1} (h - 2\, V)\, dt,$$

* For a detailed account of Hamilton's method, see Routh, Advanced Rigid Dynamics, chap. x, or Webster, Dynamics, § 41.

S may be expressed in terms of the initial and final coördinates of the system, the total energy, and the time of transit. When so expressed, Hamilton called it the *principal function*. By considering the variation produced in either of these functions by varying the final configuration of the system, Hamilton showed that from either of these functions the integrals of the differential equations of motion for the moving system could be obtained, and he discovered a partial differential equation of the first order that S would satisfy and one that A would satisfy, and so reduced the problem of solving the equations of motion for any conservative system to the solution of a partial differential equation of the first order.

It must be confessed, however, that in most cases the advantage thus gained is theoretical rather than practical, as the solving of the equation for S or for A is apt to be at least as difficult as the direct solving of the equations of motion.

CHAPTER V

APPLICATION TO PHYSICS

50. Concealed Bodies. In many problems in mechanics the configuration of the system is completely known, so that a set of coördinates can be chosen that will fix the configuration completely at any time; and the forces are given, so that if the system is conservative the potential energy can be found. In that case the Lagrangian function L or the Hamiltonian function H can be formed, and then the problem of the motion of the system can be solved completely by forming and solving the equations of motion.

In most problems in physics, however, and in some problems in mechanics the state of things is altogether different. It may be impossible to know the configuration or the forces in their entirety, so that the choice of a complete set of coördinates or the accurate forming of the potential function is beyond our powers, while it is possible to observe, to measure, and partly to control the phenomena exhibited by the moving system which we are studying. If from results which have been observed or have been deduced from experiment we are able to set up indirectly the Lagrangian function, or the Hamiltonian function, or the modified Lagrangian function, we can then form our differential equations and use them with confidence and profit.

51. Take, for instance, the motion considered in the problem of the two equal particles and the table with a hole in it (v. Art. 8, (*d*)), and suppose the investigator placed beneath the table and provided with the tools of his trade, but unable to see what is going on above the surface of the table. He sees the hanging particle and is able to determine its mass, its

velocity, and its acceleration; to fix its position of equilibrium; to measure its motion under various conditions; to apply additional forces, finite or impulsive, and to note their effects.

His system has apparently one degree of freedom. It possesses an apparent mass m and is certainly acted on by gravity with a downward force mg. He determines its position of equilibrium, and taking as his coördinate its distance x below that position, he painfully and laboriously finds that $\ddot{x}$, the acceleration of the hanging mass, is equal to

$$\frac{1}{2}\left[1-\frac{a^3}{(a-x)^3}\right]g.$$

He is now ready to call into play his knowledge of mechanics.

If T is the kinetic energy and L is the Lagrangian function for the system which he sees,

$$T=\frac{m}{2}\dot{x}^2,$$

and

$$L=T-V=\frac{m}{2}\dot{x}^2-V.$$

Since

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x},$$

$$m\ddot{x}=\frac{\partial L}{\partial x}=-\frac{\partial V}{\partial x}=\frac{m}{2}\left[1-\frac{a^3}{(a-x)^3}\right]g.$$

Therefore

$$\frac{\partial V}{\partial x}=\frac{m}{2}\left[\frac{a^3}{(a-x)^3}-1\right]g.$$

$$V=\frac{mg}{2}\left[\frac{a^3}{2(a-x)^2}-x\right]=\frac{mg}{2}\left[\frac{a^3}{2(a-x)^2}+x\right]-mgx.$$

The motion, then, can be accounted for as due to the downward force of gravity combined with a second vertical force having the potential energy $\frac{mg}{2}\left[\frac{a^3}{2(a-x)^2}+x\right]$, that is, a force vertically upward having the intensity $\frac{mg}{2}\left[\frac{a^3}{(a-x)^3}+1\right]$; and of course this force must be the pull of the string and may be due to the action of some concealed set of springs.

On the other hand, the moving system may contain some concealed body or bodies in motion, and that this is the fact is strongly suggested if a downward impulsive force P is applied to the hanging particle when at rest in its position of equilibrium. For such a force is found to impart an instantaneous velocity $\dot{x} = \frac{P}{2m}$, just half what we should get if the hanging particle were the only body in the system, and just what we should have if there were a second body of mass m above the table, connected with the hanging particle by a stretched string of fixed length.

Obviously this concealed body is *ignorable*, since we have already found a function from which we can obtain the differential equation for x, namely, the function we have just called the Lagrangian function L. It is

$$\frac{m}{2}\dot{x}^2 - \frac{mg}{2}\left[\frac{a^3}{2(a-x)^2} + x\right] + mgx$$

and contains the single coördinate x.

But if we have ignored a concealed moving body forming part of our system, this expression is not the Lagrangian function L, but is equal or proportional to Φ, the Lagrangian function modified for the coördinate or coördinates of the concealed body; and in that case some or all of the terms which we have regarded as representing the potential energy of the system may be due to the kinetic energy of the concealed body (v. Art. 41).

Of course the complete system may have two or more degrees of freedom. Let us see what we can do with two. Take x and a second coördinate θ and remember that as θ is ignorable it must be a *cyclic* coördinate and must not enter into the potential energy.

Let us now form the Lagrangian function and modify it for θ.

We have $T = A\dot{x}^2 + B\dot{x}\dot{\theta} + C\dot{\theta}^2$, where A, B, and C are functions of x.

$$p_\theta = \frac{\partial T}{\partial \dot\theta} = B\dot x + 2\,C\dot\theta.$$

$$\dot\theta = \frac{p_\theta - B\dot x}{2\,C}.$$

$$T = A\dot x^2 + \frac{B}{2\,C}(p_\theta \dot x - B\dot x^2) + \frac{(p_\theta - B\dot x)^2}{4\,C}.$$

To modify for θ we must subtract $\dot\theta p_\theta$. We get

$$M_\theta = A\dot x^2 + \frac{B}{2\,C}(p_\theta\dot x - B\dot x^2) + \frac{(p_\theta - B\dot x)^2}{4\,C} - \frac{p_\theta^{\,2} - Bp_\theta\dot x}{2\,C}$$

$$= \left[A - \frac{B^2}{4\,C}\right]\dot x^2 + \frac{B}{2\,C}p_\theta\dot x - \frac{p_\theta^{\,2}}{4\,C}.$$

Suppose that no external force acts on the concealed particle, so that the potential energy of the system is $-mgx$.

Then, if the Lagrangian function modified for θ is Φ,

$$\Phi = \left[A - \frac{B^2}{4\,C}\right]\dot x^2 + \frac{B}{2\,C}p_\theta\dot x - \frac{p_\theta^{\,2}}{4\,C} + mgx.$$

Since on our *ignoration* hypothesis θ does not enter the potential energy, the momentum $p_\theta = K$, a constant, and

$$\Phi = \left[A - \frac{B^2}{4\,C}\right]\dot x^2 + \frac{B}{2\,C}K\dot x - \frac{K^2}{4\,C} + mgx.$$

But we know that Φ is equal or proportional to

$$\frac{m}{2}\dot x^2 - \frac{mg}{2}\frac{a^3}{2(a-x)^2} + \frac{mgx}{2},$$

the function which on our hypothesis of no concealed bodies we called L.

We see that if $B = 0$, if $A = m$, and if $\dfrac{K^2}{4\,C} = \dfrac{mga^3}{2(a-x)^2}$,

whence $$C = \frac{1}{2}\frac{K^2(a-x)^2}{mga^3},$$

$$\Phi = 2\,L.$$

Then we have $$T = m\dot x^2 + \frac{1}{2}\frac{K^2(a-x)^2}{mga^3}\dot\theta^2,$$

where K is the momentum corresponding to the coördinate θ and may be any constant.

If we take for K^2 the value m^2ga^3,

$$T = \frac{m}{2}[\dot{x}^2 + \dot{x}^2 + (a-x)^2\dot{\theta}^2].$$

$$\frac{m}{2}[\dot{x}^2 + (a-x)^2\dot{\theta}^2]$$

is obviously the kinetic energy of a mass m moving in a plane and having $a - x$ and θ as polar coördinates. The Lagrangian function is equal to

$$\frac{m}{2}[2\dot{x}^2 + (a-x)^2\dot{\theta}^2] + mgx,$$

and the x Lagrangian equation is

$$2m\ddot{x} = -m(a-x)\dot{\theta}^2 + mg. \tag{1}$$

If the angular velocity of the concealed mass when the hanging particle is at rest in its position of equilibrium is $\dot{\theta}_0$, since in that case $x = 0$ and $\ddot{x} = 0$, equation (1) gives us $\dot{\theta}_0 = \sqrt{\frac{g}{a}}$. The observed motion of the hanging particle is then accounted for completely by the hypothesis that it is attached by a string to an equal particle revolving on the table and describing a circle of radius a about the hole in the table with angular velocity $\sqrt{\frac{g}{a}}$ when the hanging particle is at rest in its position of equilibrium. We see that on this hypothesis the term $\frac{mg}{2}\left[\frac{a^3}{2(a-x)^2} + x\right]$, which on the hypothesis that the system contained only the hanging particle was an unforeseen part of the potential energy, is due to the kinetic energy of the concealed moving body.

It may seem that giving K a different value might lead to a different hypothesis as to the motion of the concealed body that would account for the motion of the hanging particle.

Such, however, is not the case. We have

$$T = m\left[\dot{x}^2 + \frac{1}{2}\frac{K^2(a-x)^2}{m^2ga^3}\dot{\theta}^2\right]. \tag{2}$$

Let $$\phi = \frac{K}{m\sqrt{ga^3}}\theta.$$

Then $$T = \frac{m}{2}[\dot{x}^2 + \dot{x}^2 + (a-x)^2\dot{\phi}^2].$$

$\frac{m}{2}[\dot{x}^2 + (a-x)^2\dot{\phi}^2]$ is, as above, the kinetic energy of a body of mass m, with polar coördinates $a-x$ and ϕ; and the concealed motion is precisely as before. In using the form (2) we have merely used as our second generalized coördinate θ, a perfectly suitable parameter but one less simple than the polar angle.

52. Problems in Physics. In physical problems there may be present electrical and magnetic phenomena and concealed molecular motions, as well as the visible motions of the material parts of the system. In such cases, to fix the configuration of the system even so far as it is capable of being directly observed we must employ not only geometrical coördinates required to fix the positions of its material constituents, but also parameters that will fix its electrical or magnetic state; and as we are rarely sure of the absence of concealed molecular motions, we must often allow for the probability that the function we are trying to form by the aid of observation and experiment and on which we are to base our Lagrangian equations of motion may be the Lagrangian function modified for the ignored coördinates corresponding to the concealed molecular motions.

53. Suppose, for instance, that we have two similar, parallel, straight, conducting wires, through which electric currents due to applied electromotive forces are flowing. It is found experimentally that the wires attract each other if the currents have the same direction, and repel each other if they have opposite directions, and that reversing the currents without

altering the strength of the applied electromotive forces does not affect the observed attraction or repulsion. It is known that an electromotive force drives a current against the resistance of the conductor, and that the intensity of the current is proportional to the electromotive force. Moreover, it is known that an electromotive force does not directly cause any motion of the conductor. It is found that, as far as electric currents are concerned, the phenomena depend merely on the intensity and direction of the currents.

To fix our configuration we shall take x as the distance between the wires and take parameters to fix the intensities of the two currents. These parameters might be regarded as coördinates or as generalized velocities, but many experiments suggest that they are velocities. We shall call them $\dot{y}_1$ and $\dot{y}_2$ and define y_1 as the number of units of electricity that have crossed a right section of the first wire since a given epoch.

As all effects depend upon $\dot{y}_1$ and $\dot{y}_2$ and not on y_1 and y_2, y_1 and y_2 are *cyclic* coördinates.

Let us suppose that there are no concealed motions. Then the kinetic energy T is a homogeneous quadratic in $\dot{x}$, $\dot{y}_1$, and $\dot{y}_2$. Let

$$T = A\dot{x}^2 + L\dot{y}_1^2 + M\dot{y}_1\dot{y}_2 + N\dot{y}_2^2 + B\dot{x}\dot{y}_1 + C\dot{x}\dot{y}_2, \qquad (1)$$

where the coefficients are functions of x.

Since reversing the directions of the currents, that is, reversing the signs of $\dot{y}_1$ and $\dot{y}_2$, does not change the other phenomena, it must not affect T; therefore B and C are zero.

If the wires are not allowed to move, $\dot{x} = 0$ and T reduces to $L\dot{y}_1^2 + M\dot{y}_1\dot{y}_2 + N\dot{y}_2^2$, which is called the *electrokinetic* energy of the system. Since from considerations of symmetry this cannot be altered by interchanging the currents, $N = L$.

Let us now suppose that the first wire is fastened in position and that the only external forces are the electromotive forces E_1 and E_2, producing the currents in the two wires; the resistances $R\dot{y}_1$ and $R\dot{y}_2$ of the wires, equal, respectively, to E_1 and E_2 when the currents $\dot{y}_1$ and $\dot{y}_2$ are steady; and an ordinary

mechanical force F, tending to separate the wires. We now have

$$p_x = \frac{\partial T}{\partial \dot{x}} = 2A\dot{x},$$

and for our x Lagrangian equation

$$2A\ddot{x} + 2\frac{dA}{dx}\dot{x}^2 - \frac{dA}{dx}\dot{x}^2 - \frac{dL}{dx}\dot{y}_1^2 - \frac{dM}{dx}\dot{y}_1\dot{y}_2 - \frac{dL}{dx}\dot{y}_2^2 = F. \quad (2)$$

Let us study this equation. First suppose the electromotive forces and the impressed force F all zero, so that $\dot{y}_1 = \dot{y}_2 = 0$, and $F = 0$. Equation (2) reduces to

$$2A\ddot{x} + \frac{dA}{dx}\dot{x}^2 = 0,$$

or
$$\ddot{x} = -\frac{1}{2A}\frac{dA}{dx}\dot{x}^2.$$

If, then, a transverse velocity were impressed on the second wire, the wire would have an acceleration unless $\frac{dA}{dx} = 0$. But both wires, on our hypothesis, being inert, they can neither attract nor repel each other, therefore A is a constant; and as $T = A\dot{x}^2$ when $\dot{y}_1 = \dot{y}_2 = 0$, A is a positive constant. Therefore (2) reduces to

$$2A\ddot{x} - \frac{dL}{dx}\dot{y}_1^2 - \frac{dM}{dx}\dot{y}_1\dot{y}_2 - \frac{dL}{dx}\dot{y}_2^2 = F. \quad (3)$$

Let us now suppose that $\dot{y}_2 = 0$, and $F = 0$. Equation (3) becomes

$$2A\ddot{x} - \frac{dL}{dx}\dot{y}_1^2 = 0,$$

or
$$\ddot{x} = \frac{1}{2A}\frac{dL}{dx}\dot{y}_1^2,$$

and the second wire is attracted or repelled by the first, even when no current is flowing through the second wire. This is contrary to observation. Therefore $\frac{dL}{dx} = 0$, and L is a constant; and as $T = L\dot{y}_1^2$ when $\dot{x}$ and $\dot{y}_2$ are zero, L is a

positive constant. Equation (3) now becomes

$$2A\ddot{x} - \frac{dM}{dx}\dot{y}_1\dot{y}_2 = F,$$

and if $F = 0$,
$$\ddot{x} = \frac{1}{2A}\frac{dM}{dx}\dot{y}_1\dot{y}_2.$$

If $\dot{y}_1$ and $\dot{y}_2$ are of the same sign, $\ddot{x}$ and $\frac{dM}{dx}$ have the same sign. But according to observation the wires attract if the currents flow in the same direction, therefore $\ddot{x}$ is negative and $\frac{dM}{dx}$ is negative.

If $x = 0$, we have a single wire carrying a current of intensity $\dot{y}_1 + \dot{y}_2$. The electrokinetic energy $L\dot{y}_1^2 + M\dot{y}_1\dot{y}_2 + L\dot{y}_2^2$ becomes $L(\dot{y}_1 + \dot{y}_2)^2$ or $L\dot{y}_1^2 + 2L\dot{y}_1\dot{y}_2 + L\dot{y}_2^2$, so that the value of M when $x = 0$ is $2L$. Hence, as M is a decreasing function, M is less than $2L$ always. As M is easily seen to be zero when x is infinite, it must be positive for all values of x.

We have, then,

$$T = A\dot{x}^2 + L\dot{y}_1^2 + M\dot{y}_1\dot{y}_2 + L\dot{y}_2^2, \tag{4}$$

where A and L are positive constants and M is a positive decreasing function of x always less than $2L$. L is called the *coefficient of self-induction* of either wire per unit of length, and M the *coefficient of mutual induction* of the pair of wires per unit of length.

Our Lagrangian equations are

$$2A\ddot{x} - \frac{dM}{dx}\dot{y}_1\dot{y}_2 = F, \tag{5}$$

$$\frac{d}{dt}[2L\dot{y}_1 + M\dot{y}_2] = E_1 - R\dot{y}_1, \tag{6}$$

$$\frac{d}{dt}[2L\dot{y}_2 + M\dot{y}_1] = E_2 - R\dot{y}_2. \tag{7}$$

54. Induced Currents. (a) Suppose that no current is flowing in either of the wires considered in the last section, and that the first wire is suddenly connected with a battery furnishing an electromotive force E_1, and that thereby a current

$\dot{y}_1$ is impulsively established. Then, by Thomson's Theorem (v. Art. 30), such impulsive velocities must be set up in the system as to make the energy have the least possible value consistent with the velocity caused by the applied impulse.

If the current $\dot{y}_1$ set up in the first wire has the intensity i,

$$T = A\dot{x}^2 + Li^2 + Mi\dot{y}_2 + L\dot{y}_2^2,$$

and making T a minimum, we have

$$2A\dot{x} = 0,$$

$$Mi + 2L\dot{y}_2 = 0;$$

whence $\dot{x} = 0$, and the second wire will have no initial velocity, and $\dot{y}_2 = -\frac{Mi}{2L}$, and a current will be set up impulsively in the second wire, of intensity proportional to the intensity of the current in the first wire. Since, as we have seen, M and L are both positive, $\dot{y}_2$ is negative, and this so-called *induced* current will be opposite in direction to the impressed current $\dot{y}_1$. This impulsively induced current is soon destroyed by the resistance of the wire.

(*b*) Suppose that a steady current $\dot{y}_1$, caused by the electromotive force E_1, is flowing in the first wire while the second wire is inert, and that E_1, and consequently $\dot{y}_1$, is impulsively destroyed by suddenly disconnecting the battery. This amounts to impulsively applying to the first wire the additional electromotive force $-E_1$. If the system were initially inert, this, as we have just seen in (*a*), would immediately set up the induced current $\dot{y}_2 = \frac{Mi}{2L}$ in the second wire, and we should have $\dot{y}_1 = -i$, $\dot{y}_2 = \frac{Mi}{2L}$, as the immediate result of our impulsive action. Combine this with the initial motion in our actual problem, $\dot{y}_1 = i$, $\dot{y}_2 = 0$, and we get for our actual result $\dot{y}_1 = 0$, $\dot{y}_2 = \frac{Mi}{2L}$ (v. Art. 34).

So that if our first wire is suddenly disconnected from the battery, an induced current whose direction is the same as that of the original current is set up in the second wire. It is, however, soon destroyed by the resistance of the wire.

(c) Suppose that we have a current $\dot{y}_1$ in our fixed wire, caused by a battery of electromotive force E_1, and no current in our second wire, and that the second wire is made to move away from the first. Equations (6) and (7) of the preceding section give us

$$(4L^2-M^2)\frac{d\dot{y}_1}{dt}=2L(E_1-R\dot{y}_1)-M(E_2-R\dot{y}_2)$$
$$-(2L\dot{y}_2-M\dot{y}_1)\dot{x}\frac{dM}{dx},$$

$$(4L^2-M^2)\frac{d\dot{y}_2}{dt}=2L(E_2-R\dot{y}_2)-M(E_1-R\dot{y}_1)$$
$$-(2L\dot{y}_1-M\dot{y}_2)\dot{x}\frac{dM}{dx}.$$

When we are starting to move the second wire,

$$E_2=0,\qquad \dot{y}_2=0,\qquad E_1-R\dot{y}_1=0,$$

and we have

$$(4L^2-M^2)\frac{d\dot{y}_1}{dt}=M\dot{y}_1\dot{x}\frac{dM}{dx},$$

$$(4L^2-M^2)\frac{d\dot{y}_2}{dt}=-2L\dot{y}_1\dot{x}\frac{dM}{dx}.$$

As we have seen, L, M, $4L^2-M^2$, are positive and $\frac{dM}{dx}$ is negative. Hence the current $\dot{y}_1$ will decrease in intensity, and a current $\dot{y}_2$ having the same direction as $\dot{y}_1$ will be induced in the moving wire.

The phenomena of induced currents which we have just inferred from our Lagrangian equations are entirely confirmed by observation and experiment.

APPENDIX A

SYLLABUS. DYNAMICS OF A RIGID BODY

1. D'Alembert's Principle. In a moving system of particles the resultant of all the forces external and internal that act on any particle is called the effective force on that particle. Its rectangular components are $m\ddot{x}$, $m\ddot{y}$, and $m\ddot{z}$.

The science of rigid dynamics is based on *D'Alembert's principle*: In any moving system the actual forces impressed and internal, and the effective forces reversed in direction, form a set of forces in equilibrium, and if the system is a single rigid body, the *internal* forces are a set separately in equilibrium and may be disregarded.

It follows from this principle that in any moving system the actual forces and the effective forces are mechanically equivalent. Hence,

(*a*) $\Sigma m\ddot{x} = \Sigma X$,

(*b*) $\Sigma m\,[y\ddot{x} - x\ddot{y}] = \Sigma\,[yX - xY]$,

(*c*) $\Sigma m\,[\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z] = \Sigma\,[X\delta x + Y\delta y + Z\delta z]$.

These equations may be put into words as follows:

(*a*) The sum of those components which have a given direction is the same for the effective forces and for the actual forces.

(*b*) The sum of the moments about any fixed line is the same for the effective forces and for the actual forces.

(*c*) In any displacement of the system, actual or hypothetical, the work done by the effective forces is equal to the work done by the actual forces.

Equations (*a*) and (*b*) are called *differential equations of motion* for the system.

2. $p_x = \Sigma m v_x = \Sigma m\dot{x}$ and is the *linear momentum* of the system in the X direction; $h_z = \Sigma m\,[yv_x - xv_y] = \Sigma m\,[y\dot{x} - x\dot{y}]$ and is the *moment of momentum* about the axis of Z.

Equations (a) and (b) of § 1 may be written, respectively,

$$\frac{dp_x}{dt} = \Sigma X, \quad \text{and} \quad \frac{dh_z}{dt} = \Sigma\,[yX - xY],$$

and § 1, (a), and § 1, (b), may now be stated as follows:

(a) In a moving system the rate of change of the *linear momentum* in any given direction is equal to the sum of those components of the actual forces which have the direction in question.

(b) In a moving system the rate of change of the *moment of momentum* about any line fixed in space is equal to the sum of the moments of the actual forces about that line.

3. Center of Gravity. $$\bar{x} = \frac{\Sigma mx}{M}.$$

Hence $$p_x = \Sigma m\dot{x} = M\frac{d\bar{x}}{dt};$$

or, the linear momentum in the X direction is what it would be if the whole system were concentrated at its center of gravity.

$$h_z = M\left[\bar{y}\frac{d\bar{x}}{dt} - \bar{x}\frac{d\bar{y}}{dt}\right] + \Sigma m\left[y'\frac{dx'}{dt} - x'\frac{dy'}{dt}\right];$$

or, the *moment of momentum* about the axis of Z is what it would be if the whole mass were concentrated at the center of gravity plus what it would be if the center of gravity were at rest at the origin and the actual motion were what the relative motion about the moving center of gravity really is.

4. The *motion of the center of gravity* of a moving system is the same as if all the mass were concentrated there and all the actual forces, unchanged in direction and magnitude, were applied there.

The *motion about the center of gravity* is the same as if the center of gravity were fixed in space and the actual forces were unchanged in magnitude, direction, and point of application.

5. If the system is a rigid body containing a fixed axis,

$$h_z = Mk'^2\omega = M(h^2 + k^2)\,\omega,$$

where $Mk'^2 = M(h^2 + k^2)$ and is the *moment of inertia* about the axis, and where ω is the angular velocity of the body.

Equation (b) of § 1 becomes $Mk'^2\dfrac{d\omega}{dt} = N$, where N is the sum of the moments of the impressed forces about the fixed axis.

6. If the system is a rigid body containing a fixed point and ω_x, ω_y, ω_z, are its angular velocities about three axes fixed in space and passing through the fixed point,

$$h_z = C\omega_z - E\omega_x - D\omega_y,$$

where C is the moment of inertia about the axis of Z, and D and E are the products of inertia about the axes of X and Y, respectively; that is,

$$C = \Sigma m(x^2 + y^2), \qquad D = \Sigma myz, \qquad E = \Sigma mzx.$$

Equation (b) of § 1 becomes

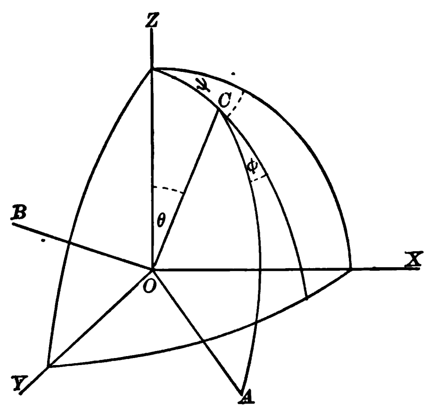
$$\frac{dh_z}{dt} = C\frac{d\omega_z}{dt} - E\frac{d\omega_x}{dt} - D\frac{d\omega_y}{dt} - (A - B)\,\omega_x\omega_y + E\omega_y\omega_z$$
$$- D\omega_z\omega_x - F(\omega_x^2 - \omega_y^2) = N,$$

where N is the sum of the moments of the impressed forces about the axis of Z.

7. Euler's Equations. If the system is a rigid body containing a fixed point and ω_1, ω_2, ω_3, are its angular velocities about the principal axes of inertia through that point (a set of axes fixed in the body and moving with it), equation (b) of § 1 becomes

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = N.$$

8. Euler's Angles. Euler's angles θ, ψ, ϕ, are coördinates of a moving system of rectangular axes A, B, C, referred to a fixed system X, Y, Z, having the same origin O. θ is the colatitude and ψ the longitude of the moving axis of C in the fixed system (regarded as a spherical system with the fixed axis of Z as the polar axis), and ϕ is the angle made by the moving CA-plane with the plane through the fixed axis of Z and the moving axis of C.



We have
$$\omega_1 = \dot\theta \sin\phi - \dot\psi \sin\theta \cos\phi,$$
$$\omega_2 = \dot\theta \cos\phi + \dot\psi \sin\theta \sin\phi,$$
$$\omega_3 = \dot\psi \cos\theta + \dot\phi;$$
and
$$\omega_x = -\dot\theta \sin\psi + \dot\phi \sin\theta \cos\psi,$$
$$\omega_y = \dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi,$$
$$\omega_z = \dot\phi \cos\theta + \dot\psi.$$

9. $\frac{1}{2} mv^2$, the *kinetic energy of a particle*, becomes

$\frac{m}{2}[\dot x^2 + \dot y^2 + \dot z^2]$ in rectangular coördinates,

$\frac{m}{2}[\dot r^2 + r^2\dot\phi^2]$ in polar coördinates,

$\frac{m}{2}[\dot r^2 + r^2(\dot\theta^2 + \sin^2\theta\dot\phi^2)]$ in spherical coördinates.

10. $\sum \frac{m}{2} v^2$, the *kinetic energy of a moving system*, becomes

$\sum \frac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2)$ in rectangular coördinates;

$\frac{M}{2} k'^2\omega^2$ if a rigid body contains a fixed axis;

$\frac{M}{2}\left[\left(\frac{d\bar x}{dt}\right)^2 + \left(\frac{d\bar y}{dt}\right)^2\right] + \frac{M}{2}k^2\omega^2$ if the body is free and the motion is two-dimensional;

$\frac{1}{2}[A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2D\omega_y\omega_z - 2E\omega_z\omega_x - 2F\omega_x\omega_y]$ if the body is rigid and contains a fixed point;

$\frac{1}{2}[A\omega_1^2 + B\omega_2^2 + C\omega_3^2]$ if the body is rigid and the axes are the principal axes for the fixed point.

11. Impulsive Forces. In a system acted on by impulsive forces, the resultant of all the impulsive forces external and internal that act on any particle is called the effective impulsive force on that particle. Its rectangular components are $m(\dot x_1 - \dot x_0)$, $m(\dot y_1 - \dot y_0)$, $(m\dot z_1 - \dot z_0)$. D'Alembert's principle holds for impulsive forces. It

follows that in any system acted on by impulsive forces, the actual forces and the effective forces are mechanically equivalent. Hence,

(*a*) $\Sigma m[\dot{x}_1 - \dot{x}_0] = \Sigma X$,

(*b*) $\Sigma m[y_1\dot{x}_1 - x_1\dot{y}_1 - y_0\dot{x}_0 + x_0\dot{y}_0] = \Sigma[yX - xY]$,

(*c*) $\Sigma m[(\dot{x}_1 - \dot{x}_0)\delta x + (\dot{y}_1 - \dot{y}_0)\delta y + (\dot{z}_1 - \dot{z}_0)\delta z]$
$= \Sigma[X\delta x + Y\delta y + Z\delta z]$.

These equations may be put into words as follows:

(*a*) The sum of the components which have a given direction is the same for the effective impulsive forces and for the actual impulsive forces.

(*b*) The sum of the moments about a given line is the same for the effective impulsive forces and for the actual impulsive forces.

(*c*) In any displacement of the system, actual or hypothetical, the sum of the virtual moments of the effective impulsive forces is equal to the sum of the virtual moments of the actual impulsive forces.

Equations (*a*) and (*b*) are the equations for the initial motion under impulsive forces. (*a*) and (*b*) may be restated as follows:

In a system acted on by impulsive forces, the total change in the linear momentum in any given direction is equal to the sum of those components of the actual impulsive forces which have the direction in question.

In a system acted on by impulsive forces, the total change in the moment of momentum about any fixed line is equal to the sum of the moments of the actual impulsive forces about that line.

Section 4 holds unaltered for impulsive forces.

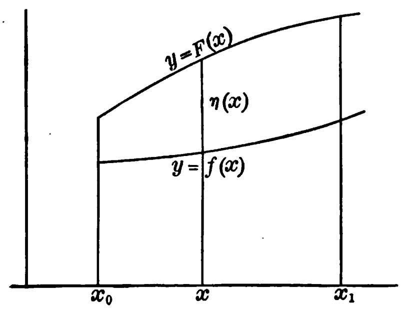
APPENDIX B

THE CALCULUS OF VARIATIONS

1. The calculus of variations owed its origin to the attempt to solve a very interesting class of problems in maxima and minima in which it is required to find the form of a function such that the definite integral of an expression involving that function and its derivatives shall be a maximum or a minimum.

Take a simple case: If $y = f(x)$, let it be required to determine the form of the function f, so that $\int_{x_0}^{x_1} \phi\left[x, y, \frac{dy}{dx}\right] dx$ shall be a maximum or a minimum.

Let $f(x)$ and $F(x)$ be two possible forms of the function. Consider their graphs $y = f(x)$ and $y = F(x)$.



If $\eta(x)$ is $F(x) - f(x)$, $\eta(x)$ can be regarded as the increment given to y by changing the form of the function from $f(x)$ to $F(x)$, the value of the independent variable x being held fast.

This increment of y, $\eta(x)$, is called the *variation* of y and is written δy; it is a function of x, and usually a wholly arbitrary function of x. The corresponding increment in y', where $y' = \frac{dy}{dx}$, can be shown to be $\eta'(x)$, and is the *variation* of y', and is written $\delta y'$, or $\delta \frac{dy}{dx}$. Obviously,

$$\delta y' = \frac{d}{dx}\delta y. \tag{1}$$

If an infinitesimal increment δy is given to y, it is proved in the differential calculus that $\frac{d}{dy}\phi(y)\delta y$ differs by an infinitesimal of higher order than δy from the increment produced in $\phi(y)$. This

approximate increment is called the variation of $\phi(y)$, so that

$$\delta\phi(y) = \frac{d}{dy}\phi(y)\,\delta y. \tag{2}$$

Similarly, $$\delta\phi(y, y') = \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta y', \tag{3}$$
or since x is not varied,

$$\delta\phi(x, y, y') = \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta y'. \tag{4}$$

As $$d\phi(y) = \frac{d}{dy}\phi(y)\,dy,$$

and $$d\phi(y, y') = \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial y'}dy',$$

we can calculate variations by the familiar formulas and processes used in calculating differentials.

2. Let α be an independent parameter. Then $y = f(x) + \alpha\eta(x)$, or $y = f(x) + \alpha\delta y$, is any one of a family of curves including $y = f(x)$ (corresponding to $\alpha = 0$) and $y = f(x) + \eta(x)$ (corresponding to $\alpha = 1$). If x_0 and x_1 are fixed values, and if

$$I(\alpha) = \int_{x_0}^{x_1}\phi(x, y + \alpha\delta y, y' + \alpha\delta y')\,dx,$$

$I(\alpha)$ is a function of the parameter α only. A necessary condition that $I(\alpha)$, a function of a single variable α, should be a maximum or a minimum when $\alpha = 0$ is $I'(\alpha) = 0$ when $\alpha = 0$.

$$I'(\alpha) = \int_{x_0}^{x_1}\frac{\partial}{\partial\alpha}\phi(x, y + \alpha\delta y, y' + \alpha\delta y')\,dx$$

$$I'(\alpha) = \int_{x_0}^{x_1}\left[\frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta y'\right]dx$$
when $\alpha = 0$.

That is, $$I'(0) = \int_{x_0}^{x_1}\delta\phi\,dx. \qquad \text{(v. § 1, (4))}$$

A necessary condition, then, that $\int_{x_0}^{x_1}\phi(x, y, y')\,dx$ should be a maximum or a minimum when $y = f(x)$ is $\int_{x_0}^{x_1}\delta\phi(x, y, y')\,dx = 0$.

$\int \delta\phi\, dx$ is taken as the definition of the variation of $\int \phi\, dx$, and our necessary condition is usually written

$$\delta \int_{x_0}^{x_1} \phi(x, y, y')\, dx = 0.$$

How this condition helps toward the determination of the form of $f(x)$ can be seen from an example.

3. Let it be required to find the form of the shortest curve $y = f(x)$ joining two given points (x_0, y_0) and (x_1, y_1). Here, since

$$ds = \sqrt{1 + y'^2}\, dx,$$

$$I = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx,$$

and I is to be made a minimum.

$$\delta I = \int_{x_0}^{x_1} \delta \sqrt{1 + y'^2} \cdot dx$$

$$= \int_{x_0}^{x_1} \frac{y' \delta y'}{\sqrt{1 + y'^2}}\, dx$$

$$= \int_{x_0}^{x_1} \frac{y' \dfrac{d}{dx} \delta y}{\sqrt{1 + y'^2}}\, dx$$

$$= \int_{x_0}^{x_1} \frac{y'}{\sqrt{1 + y'^2}}\, d\delta y.$$

Integrating by parts, this last reduces to

$$\left[\frac{y'}{\sqrt{1 + y'^2}} \delta y\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} \delta y\, dx.$$

$$\left[\frac{y'}{\sqrt{1 + y'^2}} \delta y\right]_{x_0}^{x_1} = 0,$$

since, as the ends of the path are given, $\delta y = 0$ when $x = x_0$ and when $x = x_1$. Then $\delta I = 0$ if

$$\int_{x_0}^{x_1} \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} \delta y\, dx = 0;$$

but since our δy (that is, $\eta(x)$), is a function which is wholly arbitrary,

the other factor, $\dfrac{d}{dx}\dfrac{y'}{\sqrt{1+y'^2}}$, must be equal to zero if the integral is to vanish.

This gives us

$$\frac{y'}{\sqrt{1+y'^2}} = C.$$

$$y' = c.$$

$$y = cx + d;$$

and the required curve is a straight line.

4. In our more general problem it may be shown in like manner that

$$\delta \int_{x_0}^{x_1} \phi(x, y, y')\,dx = 0$$

leads to a differential equation between y and x and so determines y as a function of x.

Of course $\delta I = 0$ is not a sufficient condition for the existence of either a maximum or a minimum and does not enable us to discriminate between maxima and minima, but like the necessary condition $\dfrac{df}{dx} = 0$ for a maximum or minimum value in a function of a single variable, it often is enough to lead us to the solution of the problem.

5. Let us now generalize a little. Let $x, y, z, \cdots$, be functions of an independent variable r, and let $x' = \dfrac{dx}{dr}$, $y' = \dfrac{dy}{dr}$, $z' = \dfrac{dz}{dr}, \cdots$. Suppose we have a function $\phi(r, x, y, z, \cdots, x', y', z', \cdots)$.

By changing the forms of the functions but holding r fast, let $x, y, z, \cdots$, be given the increments $\xi(r)$, $\eta(r)$, $\zeta(r), \cdots$.

x then becomes $x + \xi(r)$, y becomes $y + \eta(r)$, z becomes $z + \zeta(r), \cdots$; x' becomes $x' + \xi'(r)$, y' becomes $y' + \eta'(r)$, z' becomes $z' + \zeta'(r), \cdots$.

$\xi(r)$, $\xi'(r)$, are the *variations* of x and x' and are written δx and $\delta x'$. Obviously,

$$\delta x' = \frac{d}{dr}\delta x.$$

The increment produced in ϕ when infinitesimal increments δx, $\delta y, \cdots, \delta x', \delta y', \cdots$, are given to the dependent variables and to their derivatives with respect to r is known to differ from

$$\frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \cdots + \frac{\partial \phi}{\partial x'}\delta x' + \frac{\partial \phi}{\partial y'}\delta y' + \cdots$$

by terms of higher order than the variations involved. This approximate increment is called the variation of ϕ and is written $\delta\phi$. It is

found in any case precisely as $d\phi$, the complete differential of ϕ, is found.

That $\int_{r_0}^{r_1} \delta\phi(r, x, y, \cdots, x', y', \cdots) dr = 0$ is a necessary condition that $\int_{r_0}^{r_1} \phi(r, x, y, \cdots, x', y', \cdots)\, dr$ should be a maximum or a minimum can be established by the reasoning used in the case of $\int_{x_0}^{x_1} \phi(x, y, y')\, dx$. The integral $\int_{r_0}^{r_1} \delta\phi\, dr$ is called the variation of $\int_{r_0}^{r_1} \phi\, dr$, so that $\delta \int_{r_0}^{r_1} \phi\, dr = \int_{r_0}^{r_1} \delta\phi\, dr$.

6. It should be noted that our important formulas

$$\partial \frac{dx}{dr} = \frac{d}{dr}\delta x,$$

and

$$\delta \int_{r_0}^{r_1} \phi\, dr = \int_{r_0}^{r_1} \delta\phi\, dr,$$

hold only when r is the independent variable which is held fast when the forms of the functions are varied; that is, when δr is supposed to be zero.

If x and y are functions of r and we need $\delta \frac{dy}{dx}$, we get it indirectly thus:

$$\frac{dy}{dx} = \frac{y'}{x'},$$

$$\delta \frac{dy}{dx} = \delta \frac{y'}{x'} = -\frac{y'}{x'^2}\delta x' + \frac{1}{x'}\delta y' = \frac{1}{x'}\left[\frac{d}{dr}\delta y - \frac{dy}{dx}\frac{d}{dr}\delta x\right],$$

so that

$$\delta \frac{dy}{dx} = \frac{d}{dx}\delta y - \frac{dy}{dx}\frac{d}{dx}\delta x. \tag{1}$$

If we need $\delta \int \phi\, dx$, we must change our variable of integration to r.

$$\int \phi\, dx = \int \phi \frac{dx}{dr} dr.$$

$$\delta \int \phi\, dx = \delta \int \phi \frac{dx}{dr} dr.$$

$$\delta \int \phi\, dx = \int \delta\left(\phi \frac{dx}{dr}\right) dr. \tag{2}$$